Homological Algebra of Monomial Ideals

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Contents

1 Introduction **2**

2 Homological Algebra **3**
2.1 Exact Sequences and Projective and Injective Modules 3
2.2 Ext and Tor 5
2.3 Free Resolutions Over the Polynomial Ring 11

3 Monomial Ideals **13**
3.1 Introduction to Monomial Ideals 13
3.2 Simplicial Complexes 15
3.3 Graphs and Edge Ideals 16

4 Useful Applications **17**
4.1 Fröberg's Theorem 22

5 Computing $\mathbf{Ext}^{\mathbf{i}}_{\mathbf{S}}(\mathbf{S}/\mathbf{I}, \mathbf{S})$ for Monomial Ideals **24**
5.1 In the Polynomial Ring with Two Variables 26
5.2 In the Multivariate Polynomial Ring 32

1 Introduction

An important direction in commutative algebra is the study of homological invariants associated to ideals in a polynomial ring. These invariants tend to be quite mysterious for even some of the simplest ideals, such as those generated by monomials, which we address here. Let $S = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field k. A **monomial** in S is an element of the form $x_1^{a_1} \cdots x_n^{a_n}$ where each $a_i \in \mathbb{N}$ (possibly zero), and an ideal $I \subseteq S$ is a **monomial ideal** if it is generated by monomials. Although they arise most naturally in commutative algebra and algebraic geometry, monomial ideals can be further understood using techniques from combinatorics and topology. In particular, graph theory is used as a tool to study different classes of monomial ideals. Given a finite, simple graph $G = (V(G), E(G))$, we can associate to G a monomial ideal called the **edge ideal** of G, given by $I(G) = \langle x_i x_j \mid \{x_i, x_j\} \in E(G) \rangle$. This is an ideal generated by square-free monomial ideals of degree two. In fact, all such ideals come from a finite, simple graph, so we can exploit this correspondence in our study of their homological invariants.

One significant invariant of a monomial ideal I is its **minimal free resolution**, which is given by an exact complex of the following form.

$$0 \longrightarrow \bigoplus_{a \in \mathbb{N}} S(-a)^{\beta_{r,a}(I)} \longrightarrow \cdots \longrightarrow \bigoplus_{a \in \mathbb{N}} S(-a)^{\beta_{1,a}(I)} \xrightarrow{\delta} \bigoplus_{a \in \mathbb{N}} S(-a)^{\beta_{0,a}(I)} \longrightarrow I \longrightarrow 0 \qquad (1)$$

In the above complex, $\beta_{i,j}(I)$ denotes the i, j-th **Betti number** of I. These numbers are interesting because they encode important invariants of the ideal I such as the projective dimension and Castelnuovo-Mumford regularity, $\operatorname{reg}(I)$. In geometric settings, when I is an ideal defining a projective variety, the Betti numbers can be used to read off further geometric invariants such as the dimension, the degree, and the Hilbert function and polynomial. One way to measure the complexity of the resolution in (1) is to look at the degrees of the entries in the matrices of the differential maps of the resolution. The simplest situation is when all of these entries are linear forms, in which case we say that I has a **linear resolution**.

Although little is known in general about which ideals have a linear resolution, we do have a complete answer for edge ideals due to the following theorem of Fröberg.

Theorem 1. *$I(G)$ has a linear resolution if and only if the complement of G, denoted G^c, is chordal (every cycle of length > 3 has a chord).*

Herzog, Hibi, and Zheng then proved the remarkable result that all powers of ideals generated in degree two with a linear resolution also have a linear resolution. A natural question to ask next is what conditions does an edge ideal $I(G)$ need to satisfy to guarantee that $I(G)^s$ has a linear resolution for all sufficiently large s? This is currently an open question, but Nevo and Peeva have conjectured the following.

Conjecture 1. *$I(G)^s$ has a linear resolution for all $s \gg 0$ if and only if G^c has no induced 4-cycles (no cycles of length 4 that do not contain a chord).*

Working towards one direction, Nevo has proven that if G^c has no induced 4-cycles and if G satisfies an additional explicit combinatorial condition (G is *claw free*), then $I(G)^2$ has a linear resolution. However, Conjecture 1 remains open even in this case. Thus, it is an interesting question to find combinatorial conditions that guarantee higher powers of $I(G)$ have a linear resolution.

Approaching Conjecture 1 from a different direction, it is natural to study the extent to which the minimal free resolutions are linear. To formalize this notion, we use Green and Lazarsfeld's $\mathbf{N}_p$ property, which says that the first p maps in (1) are represented by matrices of linear forms. Particularly, if an edge ideal $I(G)$ satsifies the $\mathbf{N}_2$ property, i.e. the map δ in (1) is a matrix of linear forms, then we say $I(G)$ has a **linear presentation**. A theorem proven by Eisenbud, et. al gives that $I(G)$ has a linear presentation if and only if G^c has no induced 4-cycles. This result allows Conjecture 1 to be restated as follows.

Conjecture 2. *An edge ideal $I(G)$ of a finite, simple graph G has a linear presentation if and only if $I(G)^s$ has a linear resolution for all $s \gg 0$.*

This formulation has the advantage that it can be generalized by replacing $I(G)$ with an arbitrary homogeneous ideal I generated in degree two (it also makes sense for ideals generated in higher degree, but in that case, counterexamples are known to exist).

The study of monomial ideals is a popular topic of current research as there are a plethora of open questions regarding the structure of their minimal free resolutions, Betti numbers, projective dimension, regularity, etc. It is remarkable that these invariants are not well understood even for edge ideals. In this thesis we will introduce the necessary background material about homological algebra, monomial ideals, and related topics needed to understand Conjecture 2 and begin to explore new ways to attempt to prove it. While sections 2 and 3 are largely presentations of the required concepts, sections 4 and 5 contain several theorems and examples that make use of these concepts. Particularly, in section 4 we prove Fröberg's Theorem, one of the main results about edge ideals. Finally, in section 5 we show how to compute Ext groups for different classes of monomial ideals.

2 Homological Algebra

2.1 Exact Sequences and Projective and Injective Modules

We begin our study of homological algebra by introducing exact sequences, which will be used widely throughout the rest of this thesis. This section will primarily follow the material found in chapter 10.5 of [1]. Let R be a commutative ring with a unity element, denoted by 1, and let A, B, and C be R-modules. Then the pair of homomorphisms

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

is said to be **exact** (at B) if $\operatorname{im}\varphi = \ker\psi$. Similarly, a sequence of R-modules of the form

$$\cdots \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow A_{n+1} \longrightarrow \cdots$$

is called an **exact sequence** if it is exact at every A_n between a pair of homomorphisms. Such a sequence of the form

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

is called a **short exact sequence**. Using the definition of exactness, one can show that in a short exact sequence, φ is injective, ψ is surjective, and $\operatorname{im}\varphi = \ker\psi$. In this case, we say that B is an extension of C by A. This may be seen more explicitly by the common short exact sequence

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} B/A \longrightarrow 0.$$

Another important example of a short exact sequence is given by

$$0 \longrightarrow \ker\varphi \xrightarrow{\iota} F(S) \xrightarrow{\varphi} M \longrightarrow 0 \tag{2}$$

where M is an R-module, S is a set of generators for M, $F(S)$ is the free R-module on S, the homomorphism ι is the inclusion map, and φ is the unique R-module homomorphism which is the identity on S.

Given two short exact sequences of R-modules, $0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$ and $0 \longrightarrow A' \xrightarrow{\varphi'} B' \xrightarrow{\psi'} C' \longrightarrow 0$, we define a **homomorphism between short exact sequences** as a set of three R-module homomorphisms α, β, and γ that make the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d, "\alpha"] & B \arrow[r, "\psi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\varphi'"] & B' \arrow[r, "\psi'"] & C' \arrow[r] & 0.
\end{tikzcd}$$

This diagram commutes if any compositions of the homomorphisms in the diagram that start and end at the same places are equal. For example, if this this diagram commutes, then $\psi' \circ \beta \circ \varphi = \psi' \circ \varphi' \circ \alpha$ since both compositions start at A and end at C'. A useful property of a homomorphism of short exact sequences, called the Short Five Lemma [1, Ch. 10.5, Prop. 24], is that if both α and γ are injective (surjective), then β is also injective (surjective).

Given a short exact sequence of R-modules, $0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$, it is natural to study the relationship between A, B, and C with respect to different properties. For example, let D be another R-module and suppose there exists a homomorphism from D to A. Does this imply that there exists a homomorphism from D to B? We will show that the answer is yes. Let $\operatorname{Hom}_R(D, M)$ denote the set of all R-module homomorphisms from D to the R-module M. This set is actually an abelian group with composition of homomorphisms as the multiplicative law. If $f \in \operatorname{Hom}_R(D, A)$, then the composite map $f' = f \circ \varphi \in \operatorname{Hom}_R(D, B)$. Pictorially, we have the following commutative diagram:

$$\begin{tikzcd}
D \arrow[d, "f"] \arrow[dr, "f'"] & \\
A \arrow[r, "\varphi"] & B
\end{tikzcd}$$

In other words, the map $\varphi' : \operatorname{Hom}_R(D, A) \to \operatorname{Hom}_R(D, B)$ which sends $f \mapsto f' = f \circ \varphi$ is a homomorphism of abelian groups. Further, one can show that if φ is injective, then φ' is also injective. Therefore, if $0 \longrightarrow A \xrightarrow{\varphi} B$ is exact, i.e. φ is injective, then the sequence $0 \longrightarrow \operatorname{Hom}_R(D, A) \xrightarrow{\varphi'} \operatorname{Hom}_R(D, B)$ is also exact. In the language of category theory, this shows that $\operatorname{Hom}_R(D, \_)$ is a *covariant functor* which is *left exact*. This functor is said to be *exact* if

$$0 \longrightarrow \operatorname{Hom}_R(D, A) \xrightarrow{\varphi'} \operatorname{Hom}_R(D, B) \xrightarrow{\psi'} \operatorname{Hom}_R(D, C) \longrightarrow 0$$

is a short exact sequence (the map ψ' is not surjective in general). This is the case if and only if D is a **projective** R-module. For the purposes of this thesis, it is important to note that all free modules are projective. Since the functor $\operatorname{Hom}_R(D, \_)$ is not necessarily exact on the right, homological algebra is used to measure the degree to which exactness on the right fails. We will describe details of this later on in this section.

In a similar manner as above, we can instead consider the functor $\operatorname{Hom}_R(\_, D)$. Given $f \in \operatorname{Hom}_R(C, D)$, we see that the composite map $f' = f \circ \psi \in \operatorname{Hom}_R(B, D)$. Pictorially, we have the commutative diagram:

$$\begin{tikzcd} B \arrow[r, "\psi"] \arrow[dr, "f'"'] & C \arrow[d, "f"] \\ & D \end{tikzcd}$$

In other words, the map $\psi' : \operatorname{Hom}_R(C, D) \to \operatorname{Hom}_R(B, D)$ which sends $f \mapsto f' = f \circ \psi$ is a homomorphism of abelian groups. Further, one can show that if

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

is a short exact sequence of R-modules, then for any R-module D the sequence

$$0 \longrightarrow \operatorname{Hom}_R(C, D) \xrightarrow{\psi'} \operatorname{Hom}_R(B, D) \xrightarrow{\varphi'} \operatorname{Hom}_R(A, D) \tag{3}$$

is also exact, but φ' is not necessarily surjective. Therefore, $\operatorname{Hom}_R(\_, D)$ is a *contravariant functor* that is *left exact*, and is *exact* if and only if D is an **injective** R-module.

Throughout the rest of this thesis, we will frequently make use of the following statements, so we make note of them now.

Proposition 2. *Let R be a commutative ring with 1, and let M be any R-module. Then we have*

(i) $\operatorname{Hom}_R(R, M) \cong M$ *and*

(ii) $\operatorname{Hom}_R(R^n, M) \cong M^n$.

Proof. The isomorphism for (i) sends $\varphi \in \operatorname{Hom}_R(R, M)$ to $\varphi(1) = m \in M$, since φ is determined by its action on the generator of R, that is, by $\varphi(1)$.
The isomorphism for (ii) sends $\varphi \in \operatorname{Hom}_R(R^n, M)$ to the element $(\varphi(e_1), \ldots, \varphi(e_n)) \in M^n$ where $e_1, \ldots, e_n$ is a basis for R^n. $\square$

2.2 Ext and Tor

We are now ready to introduce some of the main tools used in homological algebra, as found in chapter 17 of [1]. Since this thesis will primarily focus on the properties of the homological group Ext, we will introduce this first. Then, there is a very similar, but dual, way to introduce Tor, which we will not describe in quite as much detail. To start, we consider a generalization of an exact sequence, called a cochain complex. A **cochain complex**, $\mathcal{C}^\bullet$, is a sequence of abelian group homomorphisms

$$\mathcal{C}^\bullet \ : \ 0 \longrightarrow C^0 \xrightarrow{d_1} C^1 \xrightarrow{d_2} \cdots \longrightarrow C^{n-1} \xrightarrow{d_n} C^n \xrightarrow{d_{n+1}} \cdots$$

such that $d_{n+1} \circ d_n = 0$ for each $n \in N$. Equivalently, this means that $\operatorname{im} d_n \subseteq \ker d_{n+1}$ for each n. Then the n^{th} **cohomology group** of $\mathcal{C}^\bullet$ is defined to be the quotient group

$$H^n(\mathcal{C}^\bullet) = \ker d_{n+1} / \operatorname{im} d_n.$$

The corresponding "dual" version of this involves **chain complexes**, which are descending sequences of abelian group homomorphisms

$$\mathcal{C}_\bullet \ : \ \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \longrightarrow 0$$

such that $d_{n+1} \circ d_n = 0$ for each n. Then the n^{th} **homology group** of $\mathcal{C}_\bullet$ is defined to be the quotient group

$$H_n(\mathcal{C}_\bullet) = \ker d_n / \operatorname{im} d_{n+1}.$$

We will first focus on cochain complexes in which each C^n is an R-module and each d_n is an R-module homomorphism. Then the cohomology groups are also R-modules. Additionally, observe that a cochain complex is an exact sequence if $\operatorname{im} d_n = \ker d_{n+1}$ for each n, which is equivalent to $H^n(\mathcal{C}^\bullet) = 0$ for each n. In this way, the n^{th} cohomology groups of a cochain complex measure the failure of exactness at each C^n. Similarly, the n^{th} homology groups of a chain complex measure the failure of exactness at each C_n.

As before when we defined a homomorphism between two short exact sequences, we can define a homomorphism between two cochain complexes. Let $\mathcal{A}^\bullet$ and $\mathcal{B}^\bullet$ be cochain complexes. Then a **homomorphism of complexes** $\alpha : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ is a set of homomorphism $\alpha_n : A^n \to B^n$ such that for each n the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & A^n \arrow[r] \arrow[d, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, "\alpha_{n+1}"] & \cdots \\
\cdots \arrow[r] & B^n \arrow[r] & B^{n+1} \arrow[r] & \cdots .
\end{tikzcd}$$

Using the commutativity of this diagram, one can show that a homomorphism of cochain complexes as above induces group homomorphisms between $H^n(\mathcal{A}^\bullet)$ and $H^n(\mathcal{B}^\bullet)$ for each n. Now that we have homomorphisms between complexes, we can define short exact sequences of complexes in a natural way. Let $\mathcal{A}^\bullet$, $\mathcal{B}^\bullet$, and $\mathcal{C}^\bullet$ be cochain complexes. Then a **short exact sequence of complexes**

$$0 \longrightarrow \mathcal{A}^\bullet \xrightarrow{\alpha} \mathcal{B}^\bullet \xrightarrow{\beta} \mathcal{C}^\bullet \longrightarrow 0$$

is a sequence of homomorphisms of complexes such that $0 \longrightarrow A^n \xrightarrow{\alpha_n} B^n \xrightarrow{\beta_n} C^n \longrightarrow 0$ is a short exact sequence for each n. Pictorally, we have the following diagram where every column is a short exact sequence.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A^n \arrow[r] \arrow[d, "\alpha_n"] & A^{n+1} \arrow[r] \arrow[d, "\alpha_{n+1}"] & \cdots \\
\cdots \arrow[r] & B^n \arrow[r] \arrow[d, "\beta_n"] & B^{n+1} \arrow[r] \arrow[d, "\beta_{n+1}"] & \cdots \\
\cdots \arrow[r] & C^n \arrow[r] \arrow[d] & C^{n+1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 &
\end{tikzcd}$$

We are now ready to state one of the main theorems in homological algebra.

Theorem 3 (Long Exact Sequence in Cohomology)**.** *Let* $0 \longrightarrow \mathcal{A}^\bullet \xrightarrow{\alpha} \mathcal{B}^\bullet \xrightarrow{\beta} \mathcal{C}^\bullet \longrightarrow 0$ *be a short exact sequence of cochain complexes. Then there is a long exact sequence of cohomology groups given by*

$$0 \longrightarrow H^0(\mathcal{A}^\bullet) \longrightarrow H^0(\mathcal{B}^\bullet) \longrightarrow H^0(\mathcal{C}^\bullet) \xrightarrow{\delta_0} H^1(\mathcal{A}^\bullet) \longrightarrow H^1(\mathcal{B}^\bullet) \longrightarrow H^1(\mathcal{C}^\bullet) \xrightarrow{\delta_1} H^2(\mathcal{A}^\bullet) \longrightarrow \cdots$$

where the δ_n *are called connecting homomorphisms and the maps between the respective cohomology groups are those induced by the short exact sequences*
$0 \longrightarrow A^n \xrightarrow{\alpha_n} B^n \xrightarrow{\beta_n} C^n \longrightarrow 0$ *for each* n*.*

Notice that the long exact sequence in cohomology implies that if any two of the cochain complexes are exact, i.e. have trivial cohomology groups for all n, then by exactness, the third cochain complex must also have trivial cohomology groups, so it is also exact.

In order to define Ext, we first need to understand projective resolutions. Let A be any R-module. Then A has a **projective resolution** which is an exact sequence of projective R-modules of the form

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0.$$

In particular, we can choose each P_i to be a free R-module and construct what is called a **free resolution** of A in the following way. First, let P_0 be a free R-module on a set of generators of A and define an R-module homomorphism $\epsilon : P_0 \to A$ in the natural way (as done in (2) on page 4). Then ϵ is surjective, so the sequence $P_0 \xrightarrow{\epsilon} A \longrightarrow 0$ is exact. Now we repeat this process with the submodule $\ker \epsilon \subseteq P_0$ instead of A. Let P_1 be a free R-module on a set of generators of $\ker \epsilon$ and define $d_1 : P_1 \to P_0$ such that $\operatorname{im} d_1 = \ker \epsilon$. Then the sequence $P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$ is exact. We can continue repeating this process at each step to obtain a free resolution of A. Free resolutions of modules in the multi-variable polynomial ring will be dicussed further in section three. The length of the shortest projective resolution of an R-module, which may be infinite, is called its **projective dimension**. Also, note that projective resolutions are chain complexes.

Given R-modules A and D, and a projective resolution of A, we can then apply the functor $\operatorname{Hom}_R(\_\,, D)$ to the projective resolution. Recall that this is a left exact, contravariant functor, which means that the direction of the homomorphisms is reversed, which gives the sequence

$$0 \longrightarrow \operatorname{Hom}_R(A, D) \xrightarrow{\epsilon} \operatorname{Hom}_R(P_0, D) \xrightarrow{d_1} \operatorname{Hom}_R(P_1, D) \xrightarrow{d_2} \\ \cdots \xrightarrow{d_{n-1}} \operatorname{Hom}_R(P_{n-1}, D) \xrightarrow{d_n} \operatorname{Hom}_R(P_n, D) \xrightarrow{d_{n+1}} \cdots$$

where the maps are the induced maps and we have abused notation by naming them what they were before. This is a cochain complex, so we can find its cohomology groups.

Definition 4. For the setup in the preceeding paragraph with $d_n : \operatorname{Hom}_R(P_{n+1}, D) \to \operatorname{Hom}_R(P_n, D)$, we define

$$\operatorname{Ext}^n_R(A, D) = \ker d_{n+1} / \operatorname{im} d_n$$

with $\operatorname{Ext}^0_R(A, D) = \ker d_1$ to be the n^{th} **cohomology group** derived from the functor $\operatorname{Hom}_R(\_\,, D)$.

Observe that since $P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$ is exact, the sequence

$$0 \longrightarrow \operatorname{Hom}_R(A, D) \xrightarrow{\epsilon} \operatorname{Hom}_R(P_0, D) \xrightarrow{d_1} \operatorname{Hom}_R(P_1, D)$$

is also exact. Therefore, $\operatorname{Ext}^0_R(A, D) = \ker d_1 = \operatorname{im} \epsilon \cong \operatorname{Hom}_R(A, D)$ since ϵ is injective. When computing Ext of different R-modules, we will frequently make use of this isomorphism

$$\operatorname{Ext}^0_R(A, D) \cong \operatorname{Hom}_R(A, D).$$

An important result, which we will not prove here (the details can be found in Chapter 17.1 of [1], is that the groups $\operatorname{Ext}^n_R(A, D)$ depend only on the R-modules A and D, and do not depend on the choice of projective resolution for A. Therefore, for a fixed R-module D, $\operatorname{Ext}^n_R(\_\,, D)$ is a contravariant functor from the category of R-modules to the category of abelian groups. The following theorem [1, Ch. 17.1, Thm. 8] gives a long exact sequence in cohomology coming from a short exact sequence.

Theorem 5. *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of R-modules. Then there exists a long exact sequence of abelian groups*

$$0 \longrightarrow \operatorname{Hom}_R(C, D) \longrightarrow \operatorname{Hom}_R(B, D) \longrightarrow \operatorname{Hom}_R(A, D) \xrightarrow{\delta_0} \operatorname{Ext}^1_R(C, D) \\ \longrightarrow \operatorname{Ext}^1_R(B, D) \longrightarrow \operatorname{Ext}^1_R(A, D) \xrightarrow{\delta_1} \operatorname{Ext}^2_R(C, D) \longrightarrow \cdots$$

where the maps between groups at the same level n are induced from the maps in the short exact sequence and the δ_n are connecting homomorphisms as in Theorem 3.

Observe that the first four terms of the long exact sequence in Theorem 5 are the same as the sequence given in Proposition 3, which we noted is not necessarily exact on the right. In this way, the fifth term in the exact sequence in Theorem 5, that is $\operatorname{Ext}^1_R(C, D)$, is the first measure of the failure of exactness on the right of the functor $\mathrm{Hom}_R(\_, D)$. Therefore, the sequence in Proposition 3 can be extended to a short exact sequence if and only if the connecting homomorphism δ_0 is the zero map. For examples of computing Ext groups, see Proposition 23.

We have seen above that the cohomology groups $\mathrm{Ext}^n_R(\_, D)$ determine what happens to short exact sequences on the right after applying the left exact functor $\mathrm{Hom}_R(\_, D)$. We will now show how the Tor homology groups are defined, which in a "dual" way determine what happens to short exact sequences on the left after applying the right exact functor $D \otimes_R \_$.

Let A and D be R-modules. Then $D \otimes_R A$ is an abelian group, and since R is a commutative ring, $D \otimes_R A \cong A \otimes_R D$. Then $D \otimes_R \_$ is a covariant functor that is right exact which means that given a short exact sequence of R-modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, the sequence

$$D \otimes_R A \longrightarrow D \otimes_R B \longrightarrow D \otimes_R C \longrightarrow 0$$

is an exact sequence of abelian groups, which is not necessarily exact on the left. To define Tor homology groups, we first find a projective resolution for A given by

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0.$$

Then, applying the functor $D \otimes_R \_$ to this resolution gives

$$\cdots \longrightarrow D \otimes P_n \xrightarrow{1\otimes d_n} D \otimes P_{n-1} \xrightarrow{1\otimes d_{n-1}} \cdots \xrightarrow{1\otimes d_2} D \otimes P_1 \xrightarrow{1\otimes d_1} D \otimes P_0 \xrightarrow{1\otimes \epsilon} D \otimes A \longrightarrow 0$$

which is a chain complex, so we can find its homology groups.

Definition 6. For the setup in the preceeding paragraph, we define

$$\mathrm{Tor}^R_n(D, A) = \ker(1 \otimes d_n)/\operatorname{im}(1 \otimes d_{n+1}) \qquad \text{with} \qquad \mathrm{Tor}^R_0(D, A) = D \otimes P_0/\operatorname{im}(1 \otimes d_1)$$

to be the n^{th} **homology group** derived from the functor $D \otimes_R \_$.

One can use exactness of the sequence at $D \otimes P_0$ to show that $\mathrm{Tor}^R_0(D, A) \cong D \otimes A$. It can also be shown that the homology groups $\mathrm{Tor}^R_n(D, A)$ only depend on the R-modules A and D, not on the projective resolution for A that is chosen, and that $\mathrm{Tor}^R_n(D, A) \cong \mathrm{Tor}^R_n(A, D)$. Finally, we state the analogous theorem to Theorem 5 for Tor.

Theorem 7. *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of R-modules. Then there exists a long exact sequence of abelian groups*

$$\cdots \longrightarrow \mathrm{Tor}^R_2(D, C) \xrightarrow{\delta_1} \mathrm{Tor}^R_1(D, A) \longrightarrow \mathrm{Tor}^R_1(D, B) \longrightarrow$$
$$\mathrm{Tor}^R_1(D, C) \xrightarrow{\delta_0} D \otimes A \longrightarrow D \otimes B \longrightarrow D \otimes C \longrightarrow 0$$

where the maps δ_n are called connecting homomorphisms and the maps at the same level n are induced by the short exact sequence.

Then $\mathrm{Tor}^R_1(D, C)$ is the first measure of the failure of exactness on the left of the functor $D \otimes_R \_$, and so the sequence $D \otimes_R A \longrightarrow D \otimes_R B \longrightarrow D \otimes_R C \longrightarrow 0$ can be extended to a short exact sequence if and only if δ_0 is the zero map. For an example of computing Tor homology groups, see Example 18.

Definition 8. Let $\mathcal{A}_\bullet$ be a complex of finitely generated free modules over a field, given by

$$\mathcal{A}_\bullet \;:\; 0 \longrightarrow A_n \longrightarrow \cdots \longrightarrow A_2 \longrightarrow A_1 \longrightarrow 0.$$

Then the **Euler characteristic** of the complex is $\chi(\mathcal{A}_\bullet) = \sum_{i=1}^{n} (-1)^i \dim(A_i)$.

Proposition 9. *Let $\mathcal{A}_\bullet$ be a complex as in the definition above. Then $\chi(\mathcal{A}_\bullet) = \chi(H(\mathcal{A}_\bullet))$, where $H(\mathcal{A}_\bullet)$ is the induced complex in homology given by*

$$H(\mathcal{A}_\bullet) \;:\; 0 \longrightarrow H_n(\mathcal{A}_\bullet) \xrightarrow{0} H_{n-1}(\mathcal{A}_\bullet) \xrightarrow{0} \cdots \xrightarrow{0} H_2(\mathcal{A}_\bullet) \xrightarrow{0} H_1(\mathcal{A}_\bullet) \longrightarrow 0.$$

Proof. Suppose we have

$$\mathcal{A}_\bullet \;:\; 0 \xrightarrow{\delta_{n+1}} A_n \xrightarrow{\delta_n} A_{n-1} \longrightarrow \cdots \longrightarrow A_2 \xrightarrow{\delta_2} A_1 \xrightarrow{\delta_1} 0.$$

Let $Z_i = \ker \delta_i$ for $i = 1, \ldots n$ and $B_i = \operatorname{im} \delta_{i+1}$ for $i = 0, \ldots, n$. These are commonly called cycles and boundaries, respectively. Let $a_i = \dim(A_i), b_i = \dim(B_i)$, and $z_i = \dim(Z_i)$. Observe that the First Isomorphism Theorem gives that $B_i \cong A_{i+1}/Z_{i+1}$ for $i = 0, \ldots, n-1$, so for each of these i, we have $b_i = a_{i+1} - z_{i+1}$. Then, by definition of homology groups, we have $H_i(\mathcal{A}_\bullet) = \ker \delta_i / \operatorname{im} \delta_{i+1} = Z_i/B_i$, which implies that $\dim(H_i(\mathcal{A}_\bullet)) = z_i - b_i$. Also note that $a_1 = \dim(A_1) = \dim(\ker \delta_1) = \dim(Z_1) = z_1$. Therefore, we have

$$\begin{aligned}
\chi(H(\mathcal{A}_\bullet)) &= \sum_{i=1}^{n} (-1)^i \dim(H_i(\mathcal{A}_\bullet)) \\
&= \sum_{i=1}^{n} (-1)^i (z_i - b_i) \\
&= -z_1 + \sum_{i=1}^{n-1} (-1)^{i+1} (z_{i+1} + b_i) \\
&= -a_1 + \sum_{i=1}^{n-1} (-1)^{i+1} a_{i+1} \\
&= \sum_{i=1}^{n} (-1)^i a_i = \sum_{i=1}^{n} (-1)^i \dim(A_i) = \chi(\mathcal{A}_\bullet).
\end{aligned}$$

□

Remark 10. An important fact about the Euler characteristic of a complex is that if $\mathcal{A}_\bullet$ is exact, then $\chi(\mathcal{A}_\bullet) = 0$. This follows easily from the previous proposition since $H(\mathcal{A}_\bullet) = 0$ if $\mathcal{A}_\bullet$ is exact.

2.3 Free Resolutions Over the Polynomial Ring

In this section, we will present some of the material found in the first chapter of [2]. Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field k. Let M be a finitely generated S-module. We say that M is **graded** if $M = \bigoplus_{d \in \mathbb{Z}} M_d$ where M_d is the component of M in degree d. The **Hilbert Function** of M is defined to be

$$H_M(d) = \dim_k M_d,$$

and Hilbert's idea was to compute $H_M(d)$ by using a free resolution for M. The module $M(a)$ is the module M shifted, or twisted, by a and satisfies $M(a)_d = M_{a+d}$. For example, $S(-a)$ is the free S-module of rank 1 generated by an element of degree a. The following corresponds to Proposition 2 and will also be used in later sections.

Proposition 11. *Let R be a commutative ring with 1, and let M be a graded R-module. Then for any $d, e \in \mathbb{Z}$ and any $n \in \mathbb{N}$ we have*

(i) $\mathrm{Hom}_R(R(-d), M(-e)) \cong M(d-e)$ *and*

(ii) $\mathrm{Hom}_R(R^n(-d), R(-e)) \cong R^n(d-e)$.

Proof. We will only prove (ii), since (i) follows from (ii) with $n = 1$ and $R = M$. Then for (ii), let $b_1, \ldots, b_n$ be a basis for $R^n(-d)$ such that each b_i has degree d. Then any $\varphi \in \mathrm{Hom}_R(R^n(-d), R(-e))$ is determined by its action on this basis, that is, by $\varphi(b_i) = r_i \in R(-e)$. If φ is degree-preserving, then the degree of r_i in $R(-e)$ must be d for each i. Therefore, $r_i \in R(-e)_d = R_{d-e}$ for each i. So, φ is degree-preserving if and only if $r_i \in R_{d-e}$ for each i. However, there are many homomorphisms from $R^n(-d)$ to $R(-e)$ which are not degree preserving, so in general, we get that $(\varphi(b_1), \ldots, \varphi(b_n)) = (r_1, \ldots, r_n) \in R^n(d-e)$ since each r_i is in an R-module generated in degree $d - e$. $\square$

Returning to the prior discussion with $S = k[x_1, \ldots, x_n]$, if M is a graded S-module, we can construct a graded free resolution of M in the same way described on the bottom of page 7. Let $m_i \in M$ be homogeneous elements of degree a_i that generate M as an S-module, and let $F_0 = \bigoplus_i S(-a_i)$. Then a graded free resolution of M is of the form

$$\cdots \longrightarrow F_i \xrightarrow{\varphi_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

where each F_i is a graded free module. The kernel $M_1 \subseteq F_0$ of the natural map $F_0 \to M$ is called the **first syzygy** of M. Note that we require all maps between graded modules to be degree-preserving. A remarkable fact, known as the **Hilbert Syzygy Theorem**, is that any finitely generated graded S-module M has a finite graded free resolution of length $\leq n$. In other words, the projective dimension of M is $\leq n$. This is denoted by $\mathrm{pd}(M) \leq n$, where n is the number of variables in S.

We now give a few examples of a specific kind of complex, called **Koszul complexes**, which are an important family of free resolutions.

Example 12 (Koszul Complexes)**.** The first three Koszul complexes are

(i) $\mathcal{K}_\bullet(x_1) \ : 0 \longrightarrow S(-1) \xrightarrow{(x_1)} S$

(ii) $\mathcal{K}_\bullet(x_1, x_2) \ : 0 \longrightarrow S(-2) \xrightarrow{\left(\begin{smallmatrix} x_2 \\ -x_1 \end{smallmatrix}\right)} S^2(-1) \xrightarrow{(x_1 \ x_2)} S$

(iii) $\mathcal{K}_\bullet(x_1, x_2, x_3) \ : 0 \longrightarrow S(-3) \xrightarrow{\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right)} S^3(-2) \xrightarrow{\left(\begin{smallmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{smallmatrix}\right)} S^3(-1) \xrightarrow{(x_1 \ x_2 \ x_3)} S.$

Observe that (i) is a free resolution of $S/\langle x_1 \rangle$. Similarly, one can show that (ii) is a free resolution of $S/\langle x_1, x_2 \rangle$ and (iii) is a free resolution of $S/\langle x_1, x_2, x_3 \rangle$. These complexes can be generalized to $\mathcal{K}_\bullet(x_1, \dots, x_r)$ for any $r \leq n$, giving a free resolution of $S/\langle x_1, \dots, x_r \rangle$. In particular, these complexes are important because $\mathcal{K}_\bullet(x_1, \dots, x_n)$ is a free resolution of the residue field $S/\langle x_1, \dots, x_n \rangle \cong k$ which will be used when computing other homology groups in section 4.

Every finitely generated graded S-module M has a **minimal free resolution**, which is unique up to isomorphism. This can be constructed in a similar way as a normal free resolution where at each step we choose a minimal set of generators for the kernel of the previous map. Formally, we say that a complex of graded S-modules

$$\cdots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \cdots$$

is **minimal** if for each i, $\operatorname{im} \delta_i \subseteq m \cdot F_{i-1}$, where $m = \langle x_1, \dots, x_n \rangle$ is the homogeneous maximal ideal of S. The condition of minimality is equalivalent to the condition that in a free resolution of S, none of the entries in the matrices representing the maps between the F_i's are nonzero constants. For example, the Koszul complexes in Example 12 are minimal free resolutions of $S/\langle x_1, \dots, x_r \rangle$. If M is a finitely generated S-module, then its minimal free resolution will have the form

$$0 \longrightarrow \bigoplus_{a \in \mathbb{Z}} S(-a)^{\beta_{r,a}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{a \in \mathbb{Z}} S(-a)^{\beta_{1,a}(M)} \xrightarrow{\delta_1} \bigoplus_{a \in \mathbb{Z}} S(-a)^{\beta_{0,a}(M)} \longrightarrow M \longrightarrow 0$$

and it can be shown that $\operatorname{pd}(M) = r$, the length of this resolution. In other words, the minimal free resolution of M is as short as possible. Here $\beta_{i,j}(M)$ denotes the i,j-th **Betti number** of M, which is an important invariant of M that is equal to the minimal number of generators of F_i in degree j. To compute the Betti numbers of M one often uses the following equality:

$$\beta_{i,j}(M) = \dim_k \operatorname{Tor}_i^S(k, M)_j = \dim_k \tilde{H}_i(\mathcal{K}_\bullet \otimes M)_j$$

where $\mathcal{K}_\bullet = \mathcal{K}_\bullet(x_1, \dots, x_n)$. This allows one to find the Betti numbers without actually constructing a minimal free resolution of M, which is usually very tedious. Instead, one can tensor M with the Koszul complex and take the reduced homology of the resulting complex.

Another invariant which can be defined using the Betti numbers of M is the **Castelnuovo-Mumford regularity** of M, denoted $\operatorname{reg}(M)$. This is defined as the smallest integer d such that $\beta_{i,i+d}(M) \neq 0$ for some i, but $\beta_{i,i+d+1}(M) = 0$ for all i, that is

$$\operatorname{reg}(M) = \max\{d \mid \beta_{i,i+d}(M) \neq 0 \text{ for some } i\}.$$

Another way to compute the regularity of M, which we will use several times in the next section when M is a monomial ideal, is to find the abelian groups $\operatorname{Ext}^i_S(M, S)$ and use the equality

$$\operatorname{reg}(M) = \max\{-i - j \mid \operatorname{Ext}^i_S(M, S)_j \neq 0\}.$$

Suppose M is a graded S-module that is generated in degree d. Then we say that M has a **linear resolution** if $\beta_{i,i+j}(M) = 0$ for every $j \neq d$, that is, if the only nonzero Betti numbers have the form $\beta_{i,i+d}(M)$. Further, one can show that M has a linear resolution if and only if $\operatorname{reg}(M) = d$. In this case, the maps in the minimal free resolution of M are given by matrices of linear forms. We will explore linear resolutions of monomial ideals more extensively in the remainder of this thesis.

3 Monomial Ideals

The concepts introduced in this section can be found in many sources. In the first subsection, we will focus on how monomial ideals are presented in Chapter 1 of [3] and Chapter I of [4]. In the second subsection, we will present simplicial complexes and their homology as seen in Chapter 1 of [3]. Finally, in the third subsection, we will introduce a few concepts from graph theory and edge ideals as found in Chapter 9 of [5].

3.1 Introduction to Monomial Ideals

Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field k. A **monomial** in S is an element of the form $x_1^{a_1} \cdots x_n^{a_n}$ where each $a_i \in \mathbb{N}$ (possibly zero). An ideal $I \subset S$ is a **monomial ideal** if it is generated by monomials. Additionally, a monomial is **squarefree** if each $a_i \in \{0, 1\}$, and I is a **squarefree monomial ideal** if it is generated by squarefree monomials. An important observation about monomial ideals is that I is a monomial ideal if and only if for every polynomial $f \in I$, all of the monomials in f belong to I.

Proposition 13 (Properties of Monomial Ideals)**.** *Let I and J be a monomial ideals in S. Then we have the following properties:*

(i) $I \cap J$ *is a monomial ideal,*

(ii) IJ *is a monomial ideal,*

(iii) $\sqrt{I}$ *is a monomial ideal,*

(iv) I *is a prime ideal if and only if* $I = \langle x_i \mid i \in A \rangle$ *for some* $A \subseteq \{1, \ldots, n\}$*,*

(v) I *is a radical monomial ideal, i.e.* $I = \sqrt{I}$*, if and only if* I *is a squarefree monomial ideal.*

To every monomial $x_1^{a_1} \cdots x_n^{a_n}$ in S, we can associate a unique vector $\vec{a} = (a_1, \ldots, a_n)$ in $\mathbb{N}^n$, often called the exponent vector. Let $S_{\vec{a}}$ denote the vector subspace of S spanned by $x_1^{a_1} \cdots x_n^{a_n}$, that is $S_{\vec{a}} = k\{x_1^{a_1} \cdots x_n^{a_n}\}$. Then, as a k-vector space, we can view the polynomial ring S as the direct sum of all possible $S_{\vec{a}}$ with $\vec{a} \in \mathbb{N}^n$. This gives

$$S = \bigoplus_{\vec{a} \in \mathbb{N}^n} S_{\vec{a}}$$

and since $S_{\vec{a}} \cdot S_{\vec{b}} = S_{\vec{a}+\vec{b}}$, we say that S is **multi-graded** or $\mathbb{N}^n$-graded as a k-algebra. Then monomial ideals are the $\mathbb{N}^n$-graded ideals of S, and we can write them as a direct sum, $I = \bigoplus k\{x_1^{a_1} \cdots x_n^{a_n}\}$, where the sum is over all monomials $x_1^{a_1} \cdots x_n^{a_n} \in I$. Recall that in section 2.3 we used that S is graded with respect to $\mathbb{Z}$, which is often called a "coarse" grading of S, whereas multi-grading is a "finer" grading of S. Both gradings can be used in this context, but it's important to distinguish which grading is being used.

When working with monomial ideals in S, an important consequence of the Hilbert Basis Theorem, which says that every ideal in S is finitely generated, and the grading on S is that every monomial ideal $I \subseteq S$ has a unique, finite generating set of monomials. Thus, we can write $I = \langle m_1, \ldots, m_r \rangle$ for a unique, minimal set of monomials m_i. If some positive power of each variable x_i appears in this minimal set of generators (so must have $r \geq n$), then I is said to be **artinian**. Let the degree of a monomial be given by $\deg(m) = \deg(x_1^{a_1} \cdots x_n^{a_n}) = a_1 + \cdots + a_n \in \mathbb{N}$. If all of the minimal monomial generators of I have the same degree, say d, then we say that I is generated in degree d. Throughout the remaining sections, we will frequently work with the ring S/I as both a k-vector space and an S-module. The following proposition will make it easier to compute the dimension of S/I in a given degree as a k-vector space.

Proposition 14. *[4, Prop. 1.8] Let $I \subseteq S = k[x_1, \ldots, x_n]$ be a monomial ideal. Then for every $i \in \mathbb{N}$ the k-vector space $(S/I)_i = S_i/I_i$ has a basis given by {monomials $m \in S$ such that $m \notin I$, $\deg(m) = i$}. So, $\dim_k(S/I)_i$ is equal to the number of monomials in S of degree i that are not in I.*

In order to use this proposition, it is useful to know the number of monomials in S of a certain degree, that is $\dim S_d$. A combinatorial argument shows that

$$\dim S_d = \binom{n+d-1}{d}.$$

As in section 2.3, we can construct the minimal free resolution of a monomial ideal $I \subseteq S$ and find its Betti numbers. Depending on the grading that we are using, these Betti numbers are either of the form $\beta_{i,j}(I)$ in the graded case, or $\beta_{i,\vec{a}}(I)$ where $\vec{a} \in \mathbb{N}^n$ in the multi-graded case. In both cases, the Betti numbers of S/I are the same as the Betti numbers of I shifted by one spot. This is because the minimal free resolution of I starts with $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I \longrightarrow 0$ whereas the minimal free resolution of S/I starts with $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow S \longrightarrow S/I \longrightarrow 0$. Therefore, we have

$$\beta_{i,j}(I) = \beta_{i+1,j}(S/I) \qquad \text{or} \qquad \beta_{i,\vec{a}}(I) = \beta_{i+1,\vec{a}}(S/I).$$

This implies the equality

$$\operatorname{reg}(I) = \operatorname{reg}(S/I) + 1$$

which we will use later on when computing the regularity of specific monomial ideals. Again, if I is a homogeneous monomial ideal generated in degree d, then I has a linear (minimal free) resolution if and only if $\operatorname{reg}(I) = d$.

3.2 Simplicial Complexes

We will now introduce simplicial complexes, which are closely related to squarefree monomial ideals. Let $[n] = \{1, \ldots, n\}$ represent a vertex set. A **simplicial complex** Δ on $[n]$ is a collection of subsets of $[n]$, called **faces**, which are closed under taking subsets. Therefore, if $\sigma \in \Delta$ is a face and $\tau \subseteq \sigma$, then $\tau \in \Delta$ is also a face. If $\sigma \in \Delta$ is a face with $i+1$ vertices, i.e. $|\sigma| = i+1$, then σ has dimension i and is called an i-face of Δ. The empty set $\varnothing$ is a face of every simplicial complex, and it is the only face with dimension -1. The maximal faces of a simplicial complex are called **facets** and, since faces are closed under taking subsets, a simplicial complex is determined by its facets. We are interested in simplicial complexes because they correspond to squarefree monomial ideals. To see this, we identify each face $\sigma \subseteq [n]$ with its squarefree vector, which has a 1 for each vertex in the face and a 0 otherwise. For example, $\sigma = \{2, 5, 6\} \subseteq [7]$ has squarefree vector $(0, 1, 0, 0, 1, 1, 0)$. We can then write $\mathbf{x}^\sigma = \prod_{i \in \sigma} x_i$, associating to σ a squarefree monomial. For our example we'd have $\mathbf{x}^\sigma = x_2 x_5 x_6$. Then, every simplicial complex Δ corresponds to a squarefree monomial ideal called the **Stanley-Reisner ideal** of Δ which is given by

$$I_\Delta = \langle \mathbf{x}^\tau \mid \tau \notin \Delta \rangle.$$

Therefore, the Stanley-Reisner ideal of Δ is generated by the squarefree monomials associated to the nonfaces of Δ. This correspondence between a simplicial complex and its Stanley-Reisner ideal actually creates a bijection between simplicial complexes on $[n]$ and squarefree monomial ideals in $S = k[x_1, \ldots, x_n]$.

Next, we will relate the topics from section 2 to simplicial complexes (and hence, to squarefree monomial ideals) by defining the reduced chain complex and reduced cochain complex of a simplicial complex. Let Δ be a simplicial complex on $[n]$. Let $F_i(\Delta)$ be the set of i-faces of Δ for each integer i. Then, let $k^{F_i(\Delta)}$ be a k-vector space with basis elements e_σ that correspond to the i-faces $\sigma \in F_i(\Delta)$.

Definition 15. The **reduced chain complex** of Δ over k is the complex

$$\tilde{\mathcal{C}}_\bullet(\Delta; k) \;:\; 0 \longrightarrow k^{F_{n-1}(\Delta)} \xrightarrow{d_{n-1}} \cdots \longrightarrow k^{F_i(\Delta)} \xrightarrow{d_i} k^{F_{i-1}(\Delta)} \longrightarrow \cdots \xrightarrow{d_0} k^{F_{-1}(\Delta)} \longrightarrow 0.$$

The **boundary maps** d_i are defined by setting $\operatorname{sign}(j, \sigma) = (-1)^{r-1}$ if j is the r^{th} element of the set $\sigma \subseteq \{1, \ldots, n\}$, written in increasing order, and

$$d_i(e_\sigma) = \sum_{j \in \sigma} \operatorname{sign}(j, \sigma) e_{\sigma \setminus j}.$$

This is a complex because $d_i \circ d_{i+1} = 0$ for each i, so we can find its homology groups. For each i, the k-vector space

$$\tilde{H}_i(\Delta; k) = \ker d_i / \operatorname{im} d_{i+1}$$

in **homological degree** i is the i^{th} **reduced homology** of Δ over k.

We note that the $\dim \tilde{H}_0(\Delta; k) = (\#$ of connected components of $\Delta) - 1$. We can similarly define the dual notion of the **reduced cochain complex** of Δ over k, $\tilde{\mathcal{C}}^{\bullet}(\Delta; k)$, as the vector space dual of $\tilde{\mathcal{C}}_{\bullet}(\Delta; k)$ with **coboundary maps** d^i which are the vector space duals of d_i. Then we have that

$$\tilde{H}^i(\Delta; k) = \ker d^{i+1} / \operatorname{im} d^i$$

is the i^{th} **reduced cohomology** of Δ over k. Since vector space duality preserves exact sequences, there is a canonical isomorphism $\tilde{H}^i(\Delta; k) \cong \tilde{H}_i(\Delta; k)^*$, where $(\_)^*$ denotes the vector space duality $\operatorname{Hom}_k(\_, k)$.

To each simplicial complex Δ we can associate another simplicial complex, called its **Alexander dual** $\Delta^{\vee} = \{[n] \setminus \tau \mid \tau \notin \Delta\}$, consisting of the complements of the nonfaces of Δ. We will use the Alexander dual of Δ in the next section when discussing Hochster's Formula.

3.3 Graphs and Edge Ideals

Finite, simple graphs are a specific kind of simplicial complex which have maximal facets of dimension at most 1, so everything in the previous section can be applied to graphs. We will introduce the necessary graph theory concepts and will see that finite, simple graphs are in bijection with squarefree monomial ideals generated in degree two.

Let G be a finite, simple graph (no loops or parallel edges) on $[n] = \{1, \ldots, n\}$ with edge set $E(G)$. From now on, every graph mentioned will be a finite, simple graph. If $W \subseteq [n]$ is a subset of vertices, the **induced subgraph** of G on W is denoted G_W and contains the edges $\{i, j\} \in E(G)$ with $i, j \in W$. The **complete graph** on $[n]$ is the graph G on $[n]$ that contains all possible edges, that is $\{i, j\} \in E(G)$ for all $i \neq j \in [n]$. The **complement graph** of G, denoted G^c, is the graph on $[n]$ whose edge set contains all of the non-edges of G, that is $\{i, j\} \in E(G^c)$ if and only if $\{i, j\} \notin E(G)$. A graph G is **connected** if for all $i \neq j \in [n]$ there is a sequence of edges of the form $\{\{\{i, i_2\}, \{i_2, i_3\}, \ldots, \{i_{\ell-1}, j\}\}$ between i and j. A **cycle** of G of length ℓ is a subgraph C of G such that

$$E(C) = \{\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{\ell-1}, i_\ell\}, \{i_\ell, i_1\}\},$$

where $i_1, i_2, \ldots, i_\ell$ are vertices of G such that $i_j \neq i_m$ for $j \neq m$. A **chord** of a cycle C is an edge $\{i, j\}$ of G such that i and j are vertices of C with $\{i, j\} \notin E(G)$. A graph G is **chordal** if every cycle of length greater than three in G has a chord. Every induced subgraph of a chordal graph is also chordal. A subset $W \subseteq [n]$ is a **clique** of G if $\{i, j\} \in E(G)$ for all $i \neq j \in W$. The **clique complex** of a graph G on $[n]$ is the simplicial complex $\Delta(G)$ on $[n]$ whose faces are the cliques of G.

Consider again the polynomial ring $S = k[x_1, \ldots, x_n]$ over a field k, and let G be a graph on $[n]$. As in section 3.2 above, we associate to each edge $\{i, j\} \in E(G)$ the monomial $x_i x_j$. Then the **edge ideal** of G is the squarefree monomial ideal $I(G) = \langle x_i x_j \mid \{i, j\} \in E(G)\rangle$, which is generated by the edges of G. Note that since an edge ideal $I(G)$ is generated in degree two, it will have a linear resolution if and only if $\operatorname{reg}(I(G)) = 2$. Next, consider the clique complex of G^c, $\Delta(G^c)$, whose faces are the cliques of G^c. The Stanley-Reisner ideal of $\Delta(G^c)$ is generated by the minimal nonfaces of this complex, which are exactly the edges of G. Therefore, we have that the edge ideal of G and the Stanley-Reisner ideal of the clique complex of the complement of G are the same:

$$I(G) = I_{\Delta(G^c)}.$$

Every graph on $[n]$ corresponds to a squarefree monomial ideal generated in degree two, namely its edge ideal. Conversely, any squarefree monomial ideal generated in degree two in $S = k[x_1, \ldots, x_n]$ is the edge ideal of graph on $[n]$ (take a minimal set of generators $x_i x_j$ for the ideal, each of which corresponds to an edge in the graph). Thus, there is a one-to-one correspondence between finite, simple graphs on $[n]$ and squarefree monomial ideals generated in degree two via edge ideals.

4 Useful Applications

In this section, we will describe a few useful applications of the homological algebra concepts from section 2 in studying monomial ideals and simplicial complexes. These will include Hochster's Formula for computing Tor groups of Stanley-Reisner ideals of simplicial complexes and the reduced Mayer-Vietoris exact sequence, which will both be used in the proof of Fröberg's Theorem, the main result of this section. To start, we give a formula for the dimension of the first reduced homology group of a connected graph.

Proposition 16. *Let G be a connected graph on $[n]$ with edge set $E(G)$. Then, viewing G as a one-dimensional simplicial complex over a field k, we have* $\dim_k(\tilde{H}_1(G; k)) = |E(G)| - n + 1$.

Proof. The reduced chain complex of G given by Definition 15 is

$$\tilde{\mathcal{C}}_\bullet(G; k) \ : \ 0 \longrightarrow k^{F_1(G)} \xrightarrow{d_1} k^{F_0(G)} \xrightarrow{d_0} k^{F_{-1}(G)} \longrightarrow 0$$

where $k^{F_{-1}(G)} \cong k$, $k^{F_0(G)} \cong k^n$, and $k^{F_1(G)} \cong k^{|E(G)|}$ because the empty face is the only face of dimension -1, there are n vertices, or dimension 0 faces, and there are $|E(G)|$ edges, or 1-dimensional faces. Thus, we can consider the complex

$$G_\bullet \ : \ 0 \longrightarrow k^{|E(G)|} \xrightarrow{d_1} k^n \xrightarrow{d_0} k \longrightarrow 0.$$

Taking reduced homology of this complex and using the fact that G is connected gives $\tilde{H}(G_\bullet) \ : \ 0 \longrightarrow \tilde{H}_1(G; k) \longrightarrow 0$. Then Proposition 9 applied to the the induced complex in reduced homology gives that $\chi(G_\bullet) = \chi(\tilde{H}(G_\bullet))$, where χ is the Euler characteristic.

Therefore, we have

$$(-1)\cdot 1+(-1)^2\cdot n+(-1)^3\cdot|E(G)|=(-1)\cdot\dim_k(\tilde{H}_1(G;k))$$

which implies $\dim_k(\tilde{H}_1(G;k))=|E(G)|-n+1$.
More generally, if G is not connected, then it can be shown that $\dim_k(\tilde{H}_1(G;k))=|E(G)|-n+(\#\text{ of connected components of }G)$ since $\dim\tilde{H}_0(G)+1$ is the number of connected components of G. □

Next we state Hochster's Formula and give an example to illustrate the theorem, which can be formulated in many ways and in a variety of settings. Notationally, for $\vec{a}\in\mathbb{Z}^n$, the **support** of $\vec{a}$ is denoted by $\mathrm{supp}(\vec{a})$ and is the subset of $[n]=\{1,\dots,n\}$ corresponding to the nonzero entries of $\vec{a}$. Recall that I_Δ is the Stanley-Reisner ideal of a simplicial complex, which is generated by its nonfaces. For a subset of vertices $W\subseteq[n]$, we denote the **restriction** of Δ on W by Δ_W, which contains all of the faces of Δ that consist only of vertices in W. Lastly, recall that there is a canonical isomorphism $\tilde{H}_i(\Delta;k)^*\cong\tilde{H}^i(\Delta;k)$.

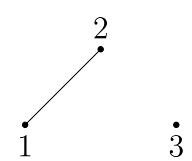
Theorem 17. *[5, Hochster's Formula, 8.1.1] Let Δ be a simplicial complex on $[n]$ and let $\vec{a}\in\mathbb{Z}^n$. Then*

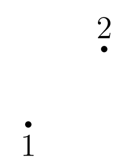
(i) $\mathrm{Tor}_i^S(k,I_\Delta)_{\vec{a}}=0$ *if $\vec{a}$ is not squarefree, and*

(ii) if $\vec{a}$ is squarefree, and $W=\mathrm{supp}(\vec{a})$, then $\mathrm{Tor}_i^S(k,I_\Delta)_{\vec{a}}\cong\tilde{H}^{|W|-i-2}(\Delta_W;k)$ *for every i.*

Using that $\beta_{i,\vec{a}}(I_\Delta)=\dim_k\mathrm{Tor}_i^S(k,I_\Delta)_{\vec{a}}$, this shows that all nonzero Betti numbers of I_Δ (and, subsequently, of S/I_Δ) lie in squarefree degrees. Although we will not present the proof of this theorem, the proof given in [5] utilizes that for a simplicial complex Δ on $[n]$, we have $\tilde{H}_{n-i-1}(\Delta;k)\cong\tilde{H}^{i-2}(\Delta^\vee;k)$ for each i, where $\Delta^\vee$ is the Alexander dual of Δ, as defined at the end of section 3.2. Consider the following example.

Example 18. Let $n=3$ and let Δ be the simplicial complex on $[3]=\{1,2,3\}$ with facets $\{1,2\}$ and $\{3\}$. Then the Alexander dual $\Delta^\vee$ consists of the complements of the minimal nonfaces of Δ. The minimal nonfaces of Δ are $\{2,3\}$ and $\{1,3\}$, so $\Delta^\vee$ has facets $\{1\}$ and $\{2\}$. Pictorally, we have

$\Delta=$ 

$\Delta^\vee=$ 

Since $n=3$, we should have $\tilde{H}_{2-i}(\Delta;k)\cong\tilde{H}^{i-2}(\Delta^\vee;k)$ for each i, and we will check that this isomorphism makes sense for this example. The reduced chain complexes of these simplicial complexes are given by

$$\begin{aligned}\tilde{\mathcal{C}}_\bullet(\Delta;k)\ &:\ 0\longrightarrow k\xrightarrow{\begin{pmatrix}1\\-1\\0\end{pmatrix}}k^3\xrightarrow{(111)}k\longrightarrow 0\\ \tilde{\mathcal{C}}_\bullet(\Delta^\vee;k)\ &:\ \qquad\quad 0\longrightarrow k^2\xrightarrow{(11)}k\longrightarrow 0.\end{aligned}$$

Computing reduced homology gives:

$$\begin{aligned}
\tilde{H}_{-1}(\Delta;k) &= k/\operatorname{im}(1\ 1\ 1) = k/k = 0 & \tilde{H}^{1}(\Delta^{\vee};k) &= 0\\
\tilde{H}_0(\Delta;k) = \frac{\operatorname{span}\{\left(\begin{smallmatrix}1\\-1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\\-1\end{smallmatrix}\right)\}}{\operatorname{span}\{\left(\begin{smallmatrix}1\\-1\\0\end{smallmatrix}\right)\}} &= \operatorname{span}\{\left(\begin{smallmatrix}0\\1\\-1\end{smallmatrix}\right)\} & \tilde{H}^{0}(\Delta^{\vee};k) &\cong \operatorname{span}\{\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)\}\\
\tilde{H}_1(\Delta;k) &= 0 & \tilde{H}^{-1}(\Delta^{\vee};k) &\cong k/k = 0\\
\tilde{H}_j(\Delta;k) &= 0,\ \text{for } j<-1, j>1 & \tilde{H}^{j}(\Delta^{\vee};k) &= 0,\ \text{for } j<-1, j>0.
\end{aligned}$$

Observe that $\dim \tilde{H}_{2-i}(\Delta;k) = \dim \tilde{H}^{i-2}(\Delta^{\vee};k)$ for each i. Also, note that we could have computed $\dim \tilde{H}_0(\Delta;k) = (\#\text{ connected components}) - 1 = 2-1 = 1$, and by the previous proposition, we have $\dim \tilde{H}_1(\Delta;k) = 1-3+2 = 0$.
Then to see how Hochster's Formula applies to this example, we consider $\mathcal{K}_\bullet(x_1,x_2,x_3)\otimes_S I_\Delta$ where $\mathcal{K}_\bullet(x_1,x_2,x_3)$, the Koszul complex, is the minimal free resolution of k and we have $I_\Delta = \langle x_1x_3, x_2x_3\rangle$. Using the multi-grading on $S = k[x_1,x_2,x_3]$, this complex is

$$0 \longrightarrow I_\Delta(-(1,1,1)) \longrightarrow \begin{matrix} I_\Delta(-(1,1,0))\\ \bigoplus\\ I_\Delta(-(0,1,1))\\ \bigoplus\\ I_\Delta(-(1,0,1))\end{matrix} \longrightarrow \begin{matrix} I_\Delta(-(1,0,0))\\ \bigoplus\\ I_\Delta(-(0,1,0))\\ \bigoplus\\ I_\Delta(-(0,0,1))\end{matrix} \longrightarrow I_\Delta \longrightarrow 0$$

Notice that taking this complex in any non-squarefree degree gives a complex that is exact. Therefore, as in (i) of Theorem 17, since Tor groups are the homology groups of this complex, we have $\operatorname{Tor}_i^S(k,I_\Delta)_{\vec{a}} = 0$ if $\vec{a}$ is not squarefree. Consider $\vec{a} = (1,1,1)$. Then $W = \{1,2,3\}$ and $\Delta_W = \Delta$, and the isomorphism shown above gives

$$\tilde{H}^{|W|-i-2}(\Delta_W;k) = \tilde{H}^{1-i}(\Delta_W;k) \cong \tilde{H}_{i-1}((\Delta_W)^{\vee};k).$$

We want to show that this is the same as $\operatorname{Tor}_i^S(k,I_\Delta)_{\vec{a}}$ for each i by showing that the complexes that these homology groups come from are essentially the same. Taking the above complex for $\mathcal{K}_\bullet(x_1,x_2,x_3)\otimes_S I_\Delta$ in degree $\vec{a}$ gives

$$0 \longrightarrow \begin{matrix} k\{x_2x_3\}\\ \bigoplus\\ k\{x_1x_3\}\end{matrix} \longrightarrow k\{x_1x_2x_3\} \longrightarrow 0.$$

Observe that this is essentially the same as the complex $\tilde{\mathcal{C}}_\bullet(\Delta^\vee; k)$ which is given by

$$0 \longrightarrow \begin{matrix} k\{x_1\} \\ \bigoplus \\ k\{x_2\} \end{matrix} \longrightarrow k \longrightarrow 0$$

and notice that the corresponding terms in these two complexes are generated by faces that are complements. Therefore, we have shown that for $\vec{a} = (1,1,1)$, we have $\operatorname{Tor}_i^S(k, I_\Delta)_{\vec{a}} \cong \tilde{H}^{|W|-i-2}(\Delta_W; k)$ for every i, which is (ii) of Theorem 17.

The following theorem will also be used to prove Fröberg's Theorem, and since it combines several of the main concepts presented in sections 2 and 3, we provide a proof as well.

Theorem 19. *[5, 5.1.8] Let Δ_1 and Δ_2 be simplicial complexes on $[n]$ over a field k. Let $\Delta = \Delta_1 \cup \Delta_2$ and $\Gamma = \Delta_1 \cap \Delta_2$. Then the* ***reduced Mayer-Vietoris sequence*** *given by*

$$\begin{aligned} \cdots \longrightarrow & \tilde{H}_i(\Gamma; k) \longrightarrow \tilde{H}_i(\Delta_1; k) \oplus \tilde{H}_i(\Delta_2; k) \longrightarrow \tilde{H}_i(\Delta; k) \\ & \longrightarrow \tilde{H}_{i-1}(\Gamma; k) \longrightarrow \tilde{H}_{i-1}(\Delta_1; k) \oplus \tilde{H}_{i-1}(\Delta_2; k) \longrightarrow \tilde{H}_{i-1}(\Delta; k) \longrightarrow \cdots \end{aligned}$$

is exact.

Proof. To prove that this is an exact sequence, we will use the theorem that is dual to Theorem 3 for reduced homology which says that every short exact sequence of complexes

$$0 \longrightarrow \mathcal{A}_\bullet \longrightarrow \mathcal{B}_\bullet \longrightarrow \mathcal{C}_\bullet \longrightarrow 0$$

gives a long exact sequence of reduced homology groups

$$0 \longrightarrow \tilde{H}_n(\mathcal{A}_\bullet) \longrightarrow \tilde{H}_n(\mathcal{B}_\bullet) \longrightarrow \tilde{H}_n(\mathcal{C}_\bullet) \longrightarrow \tilde{H}_{n-1}(\mathcal{A}_\bullet) \longrightarrow \tilde{H}_{n-1}(\mathcal{B}_\bullet) \longrightarrow \cdots$$

where n is the maximum length of $A_\bullet, B_\bullet, C_\bullet$. To do this, we need to define an appropriate short exact sequence of complexes. First, we find the reduced chain complexes of $\Delta_1, \Delta_2, \Delta$, and Γ as in Definition 15 and denote them by

$$\mathcal{A}_\bullet = \tilde{\mathcal{C}}_\bullet(\Gamma; k), \qquad \mathcal{B}_\bullet = \tilde{\mathcal{C}}_\bullet(\Delta_1; k) \oplus \tilde{\mathcal{C}}_\bullet(\Delta_2; k), \qquad \mathcal{C}_\bullet = \tilde{\mathcal{C}}_\bullet(\Delta; k).$$

Define maps between these complexes as follows:

$$\begin{aligned} \alpha_i \ &: A_i \longrightarrow B_i \\ &\quad \sigma \mapsto (\sigma, \sigma) \qquad \text{for } \sigma \in F_i(\Gamma) \text{ since } \Gamma = \Delta_1 \cap \Delta_2, \text{ and} \\ \beta_i \ &: B_i \ \longrightarrow \ C_i \\ &\quad (\sigma_1, \sigma_2) \mapsto \sigma_1 - \sigma_2 \qquad \text{for } \sigma_1 \in F_i(\Delta_1) \text{ and } \sigma_2 \in F_i(\Delta_2). \end{aligned}$$

Then for each i and every $\sigma \in F_i(\Gamma)$ we have $\beta_i \circ \alpha_i(\sigma) = \beta_i(\sigma, \sigma) = \sigma - \sigma = 0$. Therefore, $0 \longrightarrow A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \longrightarrow 0$ is a complex for each i. To show each of these complexes is exact, we need to show that α_i is injective and β_i is surjective with $\operatorname{im}\alpha_i = \ker\beta_i$. Let

$\sum_{\sigma \in F_i(\Gamma)} a_\sigma \sigma \in A_i$ with $a_\sigma \in k$. Then $\alpha_i(\sum_\sigma a_\sigma \sigma) = (\sum_\sigma a_\sigma \sigma, \sum_\sigma a_\sigma \sigma) = 0$ if and only if each term is zero, which is equivalent to the fact that $a_\sigma = 0$ for every $\sigma \in F_i(\Gamma)$, which implies $\sum_\sigma a_\sigma \sigma = 0$. Thus, α_i is injective. Now let $\sigma \in F_i(\Delta)$ and recall that $\Delta = \Delta_1 \cup \Delta_2$. If $\sigma \in F_i(\Delta_1)$, then we have $\beta_i(\sigma, 0) = \sigma$, and if $\sigma \in F_i(\Delta_2)$, then we have $\beta_i(0, -\sigma) = \sigma$. Thus, β_i is surjective. Then $\operatorname{im} \alpha_i$ is generated by all elements (σ, σ) where $\sigma \in F_i(\Gamma)$, which implies $\sigma \in F_i(\Delta_1)$ and $\sigma \in F_i(\Delta_2)$ since $\Gamma = \Delta_1 \cap \Delta_2$. Similarly, $\ker \beta_i$ is generated by all elements (σ, σ) in B_i where $\sigma \in F_i(\Delta_1)$ and $\sigma \in F_i(\Delta_2)$. Thus, $\operatorname{im} \alpha_i = \ker \beta_i$.
Since we've shown that each column in the following diagram is an exact sequence, to show that this is a short exact sequence of complexes, we need to show that this diagram commutes.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_i \arrow[r, "d_i"] \arrow[d, "\alpha_i"] & A_{i-1} \arrow[r] \arrow[d, "\alpha_{i-1}"] & \cdots \\
\cdots \arrow[r] & B_i \arrow[r, "d_i"] \arrow[d, "\beta_i"] & B_{i-1} \arrow[r] \arrow[d, "\beta_{i-1}"] & \cdots \\
\cdots \arrow[r] & C_i \arrow[r, "d_i"] \arrow[d] & C_{i-1} \arrow[r] \arrow[d] & \cdots \\
& 0 & 0 &
\end{tikzcd}$$

To show that the top square commutes, let $\sigma = (n_1, \ldots, n_i) \in F_i(\Gamma)$. Then

$$\begin{aligned}
\alpha_{i-1} \circ d_i(\sigma) &= \alpha_{i-1}\left(\sum_j (-1)^j (n_1, \ldots, \widehat{n_j}, \ldots, n_i)\right) \\
&= \left(\sum_j (-1)^j (n_1, \ldots, \widehat{n_j}, \ldots, n_i), \sum_j (-1)^j (n_1, \ldots, \widehat{n_j}, \ldots, n_i)\right) \quad \text{and} \\
d_i \circ \alpha_i(\sigma) &= d_i(\sigma, \sigma) = (d_i(\sigma), d_i(\sigma)) \\
&= \left(\sum_j (-1)^j (n_1, \ldots, \widehat{n_j}, \ldots, n_i), \sum_j (-1)^j (n_1, \ldots, \widehat{n_j}, \ldots, n_i)\right).
\end{aligned}$$

Thus, $\alpha_{i-1} \circ d_i = d_i \circ \alpha_i$, so the top square commutes.
For the bottom square, then $\sigma_1 \in F_i(\Delta_1)$ and $\sigma_2 \in F_i(\Delta_2)$. Then $(\sigma_1, \sigma_2) \in B_i$ and we have

$$\begin{aligned}
\beta_{i-1} \circ d_i(\sigma_1, \sigma_2) &= \beta_{i-1}(d_i(\sigma_1), d_i(\sigma_2)) = d_i(\sigma_1) - d_i(\sigma_2) \quad \text{and} \\
d_i \circ \beta_i(\sigma_1, \sigma_2) &= d_i(\sigma_1 - \sigma_2) = d_i(\sigma_1) - d_i(\sigma_2).
\end{aligned}$$

Thus, $\beta_{i-1} \circ d_i = d_i \circ \beta_i$, so the bottom square commutes.

Therefore, we have constructed a short exact sequence of complexes

$$0 \longrightarrow \mathcal{A}_\bullet \xrightarrow{\alpha} \mathcal{B}_\bullet \xrightarrow{\beta} \mathcal{C}_\bullet \longrightarrow 0$$

which gives a long exact sequence of reduced homology groups

$$\cdots \longrightarrow \tilde{H}_i(\mathcal{A}_\bullet) \longrightarrow \tilde{H}_i(\mathcal{B}_\bullet) \longrightarrow \tilde{H}_i(\mathcal{C}_\bullet) \longrightarrow \tilde{H}_{i-1}(\mathcal{A}_\bullet) \longrightarrow \tilde{H}_{i-1}(\mathcal{B}_\bullet) \longrightarrow \cdots .$$

By our construction, this is precisely the Mayer-Vietoris sequence

$$\begin{aligned} \cdots \longrightarrow & \tilde{H}_i(\Gamma; k) \longrightarrow \tilde{H}_i(\Delta_1; k) \oplus \tilde{H}_i(\Delta_2; k) \longrightarrow \tilde{H}_i(\Delta; k) \\ & \longrightarrow \tilde{H}_{i-1}(\Gamma; k) \longrightarrow \tilde{H}_{i-1}(\Delta_1; k) \oplus \tilde{H}_{i-1}(\Delta_2; k) \longrightarrow \tilde{H}_{i-1}(\Delta; k) \longrightarrow \cdots \end{aligned}$$

so it must also be exact. □

4.1 Fröberg's Theorem

We now have the necessary tools to prove Fröberg's Theorem, which characterizes all edge ideals with linear resolution by a remarkably simple combinatorial property. In fact, this theorem says precisely when a given squarefree monomial ideal generated in degree two has a linear resolution, by the correspondence described at the end of section 3.3. We will present the proof of this theorem that is given in chapter 9.2 of [5].

Let G be a graph on $[n]$. Then we say that G is **decomposable** if there exist proper subsets P and Q of $[n]$ with $P \cup Q = [n]$ such that $P \cap Q$ is a clique of G and $\{i, j\} \notin E(G)$ for every $i \in P \setminus Q$ and every $j \in Q \setminus P$. Recall that a chordal graph is one in which every cycle of length greater than 3 has a chord.

Lemma 20. *Every chordal graph which is not complete is decomposable.*

We will not give the proof of this lemma, which is entirely graph-theoretic and given in full detail in Lemma 9.2.1 of [5]. However, we will give the proof of the following corollary, and will then describe an example to illustrate the proof.

Corollary 21. *Let G be a chordal graph on $[n]$ and let $\Delta(G)$ be its clique complex. Then $\tilde{H}_i(\Delta(G); k) = 0$ for every $i \neq 0$.*

Proof. We will show that $\tilde{H}_i(\Delta(G); k) = 0$ for every $i \neq 0$ by induction on the number of vertices n. First, suppose $n = 1$. Then $G = \Delta(G) = \{1\}$ is just a point, so $\tilde{H}_i(\Delta(G); k) = 0$ for every i. Suppose that $\tilde{H}_i(\Delta(G); k) = 0$ for every $i \neq 0$ holds for graphs on $[n-1]$, and let G be a chordal graph on $[n]$. If G is a complete graph, then $\Delta(G)$ is a n-simplex. Thus, $\tilde{H}_i(\Delta(G); k) = 0$ for every i (this is a standard result about simplices, see [5, Example 5.1.9]). So, suppose G is not complete. Then by the above lemma, G is decomposable, so there exist proper subsets P and Q of $[n]$ such that $P \cup Q = [n]$, $P \cap Q$ is a clique of G, and for every $i \in P \setminus Q$ and every $j \in Q \setminus P$ we have $\{i, j\} \notin E(G)$. Let

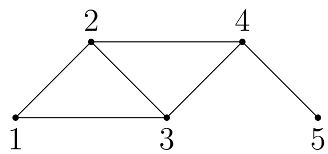
$$\Delta = \Delta(G), \quad \Delta_1 = \Delta(G_P), \quad \Delta_2 = \Delta(G_Q), \quad \text{and} \quad \Gamma = \Delta(G_{P \cap Q}).$$

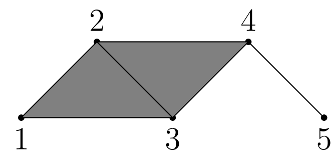
Then $\Delta = \Delta_1 \cup \Delta_2$ and $\Gamma = \Delta_1 \cap \Delta_2$. Since G is chordal, the induced subgraphs G_P and G_Q are also chordal. Each of G_P and G_Q has strictly less than n vertices since P and Q are proper subsets, so the inductive assumption gives that $\tilde{H}_i(\Delta_1; k) = 0$ and $\tilde{H}_i(\Delta_2; k) = 0$ for every $i \neq 0$. Since $P \cap Q$ is a clique of G, we know that Γ is a simplex, so $\tilde{H}_i(\Gamma; k) = 0$ for every i. Then Theorem 19 gives that the Mayer-Vietoris sequence

$$\cdots \longrightarrow \tilde{H}_i(\Gamma; k) \longrightarrow \tilde{H}_i(\Delta_1; k) \oplus \tilde{H}_i(\Delta_2; k) \longrightarrow \tilde{H}_i(\Delta; k) \longrightarrow \tilde{H}_{i-1}(\Gamma; k) \longrightarrow \cdots$$

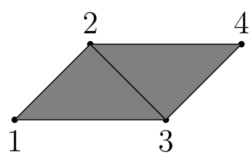
is exact for $i \neq 0$. Hence, $\tilde{H}_i(\Delta; k) = 0$ for every $i \neq 0$. $\square$

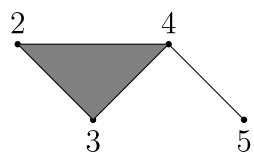
Consider the following example for $n = 5$. Let G be a graph on $[5]$ with edges $12, 13, 23, 24, 34$, and 45. Then $\Delta(G)$ has facets $123, 234$, and 45. Pictorally, we have

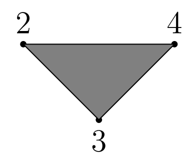
$G =$ 

$\Delta(G) =$ 

Then G is a chordal graph that is not complete, so we know it is decomposable by the lemma. Let $P = \{1, 2, 3, 4\}$ and $Q = \{2, 3, 4, 5\}$. These are proper subsets whose union is all of the vertices and whose intersection is $\{2, 3, 4\}$, which is a clique of G. We also have that 15 is not an edge of G. As in the proof of corollary, we have

$\Delta_1 =$ 

$\Delta_2 =$ 

$\Gamma =$ 

Observe that we do indeed have $\Delta = \Delta_1 \cup \Delta_2$ and $\Gamma = \Delta_1 \cap \Delta_2$. Thus, we can use the Mayer-Vietoris exact sequence to see that $\tilde{H}_i(\Delta(G); k) = 0$ for all $i \neq 0$.
We will now prove the main result of this section using the previous lemma and corollary.

Theorem 22 (Fröberg's Theorem)**.** *The edge ideal $I(G)$ of a finite graph G has a linear resolution if and only if G^c is chordal.*

Proof. Let G be a graph on $[n]$. As described in section 3.3, $I(G) = I_{\Delta(G^c)}$, so it suffices to show that $I_{\Delta(G^c)}$ has a linear resolution if and only if G^c is chordal. We know that $I_{\Delta(G^c)}$ has a linear resolution if the only nonzero Betti numbers are of the form $\beta_{i,i+2}(I_{\Delta(G^c)})$. We also know that $\beta_{i,j}(I_{\Delta(G^c)}) = \dim \operatorname{Tor}_i^S(k, I_{\Delta(G^c)})_j$, and Hochster's formula (Theorem 17) gives that $\operatorname{Tor}_i^S(k, I_{\Delta(G^c)})_{|W|} \cong \tilde{H}^{|W|-i-2}(\Delta(G^c)_W; k)$ for each i when $W = \operatorname{supp}(\vec{a})$ with $\vec{a} \in \mathbb{Z}^n$ squarefree. Therefore, $\beta_{i,|W|}(I_{\Delta(G^c)}) = \dim \tilde{H}^{|W|-i-2}(\Delta(G^c)_W; k)$. This implies that $I_{\Delta(G^c)}$ has a linear resolution if and only if $\tilde{H}^i(\Delta(G^c)_W; k) = 0$ for all $i \neq 0$ (that is, $|W| \neq i + 2$) and all subsets $W \subseteq [n]$.
Suppose G^c is not chordal. Then there exists a cycle C of length greater than 3 which does not have a chord. Let W be the vertices of C and say $|W| = \ell \geq 4$. Since C is a cycle with no chord, it also has ℓ edges. Then Proposition 16 gives that $\dim_k(\tilde{H}_1(C; k)) = \ell - \ell + 1 = 1$. But $\Delta(G^c)_W$ is exactly the same as C, so we have $\tilde{H}^1(\Delta(G^c)_W; k) \cong \tilde{H}_1(C; k) \neq 0$. Thus,

$I_{\Delta(G^c)}$ does not have a linear resolution.
Now suppose that G^c is chordal and let $W \subseteq [n]$ be arbitrary. Then $\Delta(G^c)_W$ is the clique complex of restriction of G^c to W, i.e. G^c_W, which is also chordal. The corollary then gives that $\tilde{H}_i(\Delta(G^c)_W; k) = 0$ for every $i \neq 0$. Since $\tilde{H}_i(\Delta(G^c)_W; k) \cong \tilde{H}^i(\Delta(G^c)_W; k)$, this implies that $I_{\Delta(G^c)}$ has a linear resolution. $\square$

5 Computing $\mathbf{Ext^i_S(S/I, S)}$ for Monomial Ideals

First, we show how to find the cohomology groups $\mathrm{Ext}^i_S(M, S)$ where M is one of a few specific S-modules. Recall that for $a \in \mathbb{Z}$, $k(a)$ is a copy of k generated in degree $-a$.

Proposition 23. *Let $S = k[x_1, \ldots, x_n]$ and let $m = \langle x_1, \ldots, x_n \rangle$. Then we have*

(i) $\mathrm{Ext}^i_S(S/m, S) = \begin{cases} 0, & i \neq n \\ k(n), & i = n \end{cases}.$

(ii) For $j \geq 1$, $\mathrm{Ext}^i_S(m^j/m^{j+1}, S) = \begin{cases} 0, & i \neq n \\ \bigoplus\limits_{1}^{\binom{n+j-1}{j}} k(n+j), & i = n \end{cases}.$

(iii) For $d \geq 1$, $\mathrm{Ext}^i_S(S/m^d, S) = \begin{cases} 0, & i \neq n \\ k(n) + \bigoplus\limits_{j=1}^{d-1} \bigoplus\limits_{1}^{\binom{n+j-1}{j}} k(n+j), & i = n \end{cases}.$

Proof. For (i), we apply the left exact functor $\mathrm{Hom}_S(\_, S)$ to a projective resolution of S/m. We will use the Koszul complex $\mathcal{K}_\bullet(x_1, \ldots, x_n)$ (see Example 12) which is a minimal free resolution of S/m. Then $\mathrm{Ext}^i_S(S/m, S)$ is the i^{th} cohomology group derived from the functor $\mathrm{Hom}_S(\_, S)$, and since $\mathrm{Hom}_S(\mathcal{K}_\bullet(x_1, \ldots, x_n), S)$ is also a Koszul complex up to a degree shift, all of these cohomology groups will be zero for $i \leq n-1$. For $i = n$, using Proposition 11, we have $\mathrm{Ext}^n_S(S/m, S) = \mathrm{Hom}_S(S(-n), S)/m \cong S(n)/m = k(n)$.
For (ii), we use that as an S-module we have

$$m^j/m^{j+1} = \frac{\langle x_1, \ldots, x_n \rangle^j}{\langle x_1, \ldots, x_n \rangle^{j+1}} \cong S/m(-j)^{\binom{n+j-1}{j}}$$

where we have used that $\dim S_j = \binom{n+j-1}{j}$ (see page 14). Then using that Ext commutes with finite direct sums gives

$$\mathrm{Ext}^i_S((m^j/m^{j+1}, S) = \mathrm{Ext}^i_S(S/m(-j)^{\binom{n+j-1}{j}}, S) = \bigoplus_{1}^{\binom{n+j-1}{j}} \mathrm{Ext}^i_S(S/m(-j), S).$$

By part (i) and Proposition 11 we have $\mathrm{Ext}^i_S(S/m(-j), S) = 0$ for $i \leq n-1$ and for $i = n$ we have $\mathrm{Ext}^n_S(S/m(-j), S) = k(n+j)$.

For (iii), consider the filtration $0 \subseteq m^{d-1}/m^d \subseteq \cdots \subseteq m^2/m^d \subseteq m/m^d \subseteq S/m^d$. Then we have the short exact sequence

$$0 \longrightarrow m/m^d \longrightarrow S/m^d \longrightarrow S/m \longrightarrow 0,$$

and we know Ext of the last term, so we find a short exact sequence with the first term

$$0 \longrightarrow m^2/m^d \longrightarrow m/m^d \longrightarrow m/m^2 \longrightarrow 0,$$

and continue this process until we get

$$0 \longrightarrow m^{d-1}/m^d \longrightarrow m^{d-2}/m^d \longrightarrow m^{d-2}/m^{d-1} \longrightarrow 0$$

at which point we know Ext of each term in the complex except the middle term by part (ii). The second through last short exact sequences give

$$\begin{aligned}
\operatorname{Ext}^i_S(m/m^d, S) &= \operatorname{Ext}^i_S(m/m^2, S) \oplus \operatorname{Ext}^i_S(m^2/m^3, S) \oplus \cdots \oplus \operatorname{Ext}^i_S(m^{d-1}/m^d, S) \\
&= \bigoplus_{j=1}^{d-1} \operatorname{Ext}^i_S(m^j/m^{j+1}, S) \\
&= \begin{cases} 0, & i \neq n \\ \displaystyle\bigoplus_{j=1}^{d-1} \bigoplus_{1}^{\binom{n+j-1}{j}} k(n+j), & i = n. \end{cases}
\end{aligned}$$

Then taking the long exact sequence in cohomology coming from the first short exact sequence (Theorem 5) gives that $\operatorname{Ext}^i_S(S/m^d, S) = 0$ for all $i \leq n-1$ and for $i = n$ we have the short exact sequence

$$0 \longrightarrow \operatorname{Ext}^n_S(S/m, S) \longrightarrow \operatorname{Ext}^n_S(S/m^d, S) \longrightarrow \operatorname{Ext}^n_S(m/m^d, S) \longrightarrow 0.$$

Thus, we have

$$\operatorname{Ext}^n_S(S/m^d, S) = \operatorname{Ext}^n_S(S/m, S) + \operatorname{Ext}^n_S(m/m^d, S) = k(n) + \bigoplus_{j=1}^{d-1} \bigoplus_{1}^{\binom{n+j-1}{j}} k(n+j).$$

□

It is worth noting that the third part of the above proposition proves that m^d has a linear resolution for all $d \geq 1$. Indeed, we have

$$\begin{aligned}
\operatorname{reg}(m^d) = \operatorname{reg}(S/m^d) + 1 &= \max\{-i-j \mid \operatorname{Ext}^i_S(S/m^d, S)_j \neq 0\} + 1 \\
&= \max\{n-n, n+1-n, \ldots, n+d-1-n\} + 1 = d
\end{aligned}$$

and since m^d is a homogeneous ideal generated in degree d with $\operatorname{reg}(m^d) = d$, it has a linear resolution.

5.1 In the Polynomial Ring with Two Variables

Let $S = k[x, y]$ be the polynomial ring in two variables over a field, k, and let I be an arbitrary monomial ideal in S. In this section, we will show how to compute $\mathrm{Ext}^i_S(S/I, S)$ by considering two cases: when I is artinian, and when I is not artinian.

Case 1: I is artinian

If I is artinian, then we can find a minimal set of monomials that generate I which includes some power of x and some power of y, so that we get

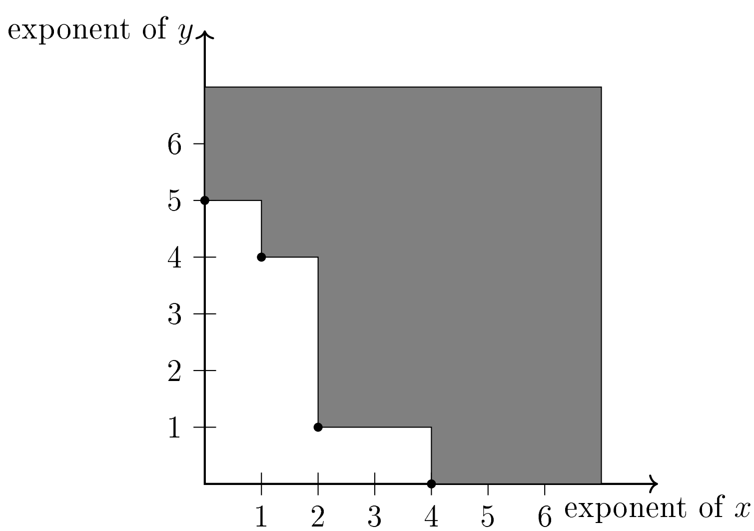
$$I = \langle x^{p_1}, y^{p_2}, m_1, \ldots, m_r\rangle \qquad \text{for some} \qquad p_1, p_2, r \geq 0,\ m_j = x^{\alpha_j} y^{\beta_j} \in S.$$

If $I = S$, then $I = \langle 1\rangle$, and $\mathrm{Ext}^i_S(S/I, S) = \mathrm{Ext}^i_S(0, S) = 0$ for all i.
So, suppose that $I \neq S$, then S/I is nonzero and we have $p_1, p_2 \geq 1$ and $\alpha_j, \beta_j \geq 1$ for each j (if $r \geq 1$). In order to visualize I as a subset of $\mathbb{R}^2$, consider the set of points

$$A = \{(p_1, 0), (0, p_2)\} \cup \{(\alpha_j, \beta_j) \mid m_j = x^{\alpha_j} y^{\beta_j}, j = 1, \ldots, r\}.$$

This set contains the exponent vectors of the minimal generating set of I. Plotting these points in the first quadrant of $\mathbb{R}^2$, we can see that the points in A outline a staircase. The ideal I can then be visualized as the set containing this staircase and all of the points above and to the right of the staircase. We illlustrate this idea with an example.

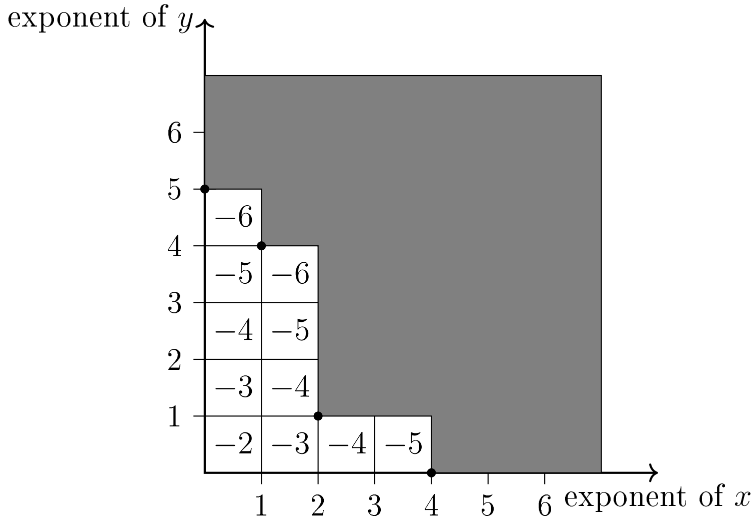
Example 24. Let $I = \langle x^4, y^5, xy^4, x^2y\rangle \subseteq S$. Then $A = \{(4, 0), (0, 5), (1, 4), (2, 1)\}$ and we can visualize I in $\mathbb{R}^2$ as the shaded region below which extends upward and to the right.



We can then easily visualize S/I as the part of the first quadrant not contained in I. Since we are only concerned with points having integer coordinates (as these are the only possible exponent vectors), consider these points in S/I and let each point with both coordinates nonzero represent the unit square in the plane of which it is the upper-right corner. To each of these squares in S/I we assign a value in the following way:

Let (α, β) be the upper-right corner of a unit square in S/I. Then the value of that square is $-(\alpha+\beta)$.

For our example, we have



Observe that since I is artinian, there are only finitely many integer points in S/I, and hence, finitely many unit squares in S/I. We are now ready to compute $\operatorname{Ext}^i_S(S/I, S)$ in this case.

Proposition 25. *Let I be an artinian monomial ideal in $S = k[x, y]$, and write $I = \langle x^{p_1}, y^{p_2}, m_1, \ldots, m_r\rangle$ as above. Denote the values of each of the unit squares in S/I by $d_1, \ldots, d_n$. Then we have*

$$\operatorname{Ext}^i_S(S/I, S) = \begin{cases} k(-d_1) \oplus k(-d_2) \oplus \cdots \oplus k(-d_n), & i = 2 \\ 0, & i \neq 2. \end{cases}$$

Therefore, each unit square in S/I corresponds to a copy of k generated in degree d_j in $\operatorname{Ext}^2_S(S/I, S)$.

Proof. We will prove this by induction on the number of unit squares in S/I. If there is one square, then we must have $I = \langle x, y\rangle = m$ and this square has value $d_1 = -2$. By Proposition 23, we have

$$\operatorname{Ext}^i_S(S/m, S) = \begin{cases} k(2), & i = 2 \\ 0, & i \neq 2 \end{cases}$$

so the proposition holds in this case.

Now suppose that S/I has n unit squares with values given by $d_1, \ldots, d_n$. Without loss of generality, we can assume that $|d_n| \geq |d_j|$ for each $j = 1, \ldots, n$. Let J be the ideal containing I and the unit square with value d_n. Then S/J has $n-1$ unit squares with values $d_1, \ldots, d_{n-1}$, so the inductive assumption applies. We have the following short exact sequence

$$0 \longrightarrow S/J \longrightarrow S/I \longrightarrow J/I \longrightarrow 0$$

which (by Theorem 5) gives the long exact sequence in cohomology

$$\begin{aligned}0 \longrightarrow \operatorname{Ext}^0_S(J/I,S) \longrightarrow \operatorname{Ext}^0_S(S/I,S) \longrightarrow \operatorname{Ext}^0_S(S/J,S)\\ \longrightarrow \operatorname{Ext}^1_S(J/I,S) \longrightarrow \operatorname{Ext}^1_S(S/I,S) \longrightarrow \operatorname{Ext}^1_S(S/J,S)\\ \longrightarrow \operatorname{Ext}^2_S(J/I,S) \longrightarrow \operatorname{Ext}^2_S(S/I,S) \longrightarrow \operatorname{Ext}^2_S(S/J,S) \longrightarrow 0.\end{aligned}$$

Consider that $J/I \cong$ (vector space generated by unit square with value d_n) $\cong k \cdot x^\alpha y^\beta$ where $d_n = -(\alpha+1+\beta+1) = -(\alpha+\beta+2)$, i.e. (α,β) is the lower-left corner of the square. Then the S-module homomorphism $\varphi : S(d_n+2) \longrightarrow k \cdot x^\alpha y^\beta$ such that $1 \mapsto x^\alpha y^\beta$ is surjective and has $\ker\varphi = \langle x,y\rangle = m$. Thus, the first isomorphism theorem gives that $J/I \cong k \cdot x^\alpha y^\beta \cong S(d_n+2)/m$ as an S-module. This gives

$$\operatorname{Ext}^i_S(J/I,S) = \operatorname{Ext}^i_S(S(d_n+2)/m,S) = \begin{cases} k(2-(d_n+2)) = k(-d_n), & i=2\\ 0, & i\neq 2\end{cases}.$$

Using this and the inductive assumption, the long exact sequence then becomes

$$\begin{aligned}0 \longrightarrow 0 \longrightarrow \operatorname{Ext}^0_S(S/I,S) \longrightarrow 0\\ \longrightarrow 0 \longrightarrow \operatorname{Ext}^1_S(S/I,S) \longrightarrow 0\\ \longrightarrow k(-d_n) \longrightarrow \operatorname{Ext}^2_S(S/I,S) \longrightarrow k(-d_1)\oplus\cdots\oplus k(-d_{n+1}) \longrightarrow 0.\end{aligned}$$

Exactness then gives that

$$\operatorname{Ext}^i_S(S/I,S) = \begin{cases} k(-d_1)\oplus\cdots\oplus k(-d_{n-1})\oplus k(-d_n), & i=2\\ 0, & i\neq 2\end{cases}$$

which completes the proof. □

Now, we will consider monomial ideals in $S = k[x,y]$ which are not artinian.

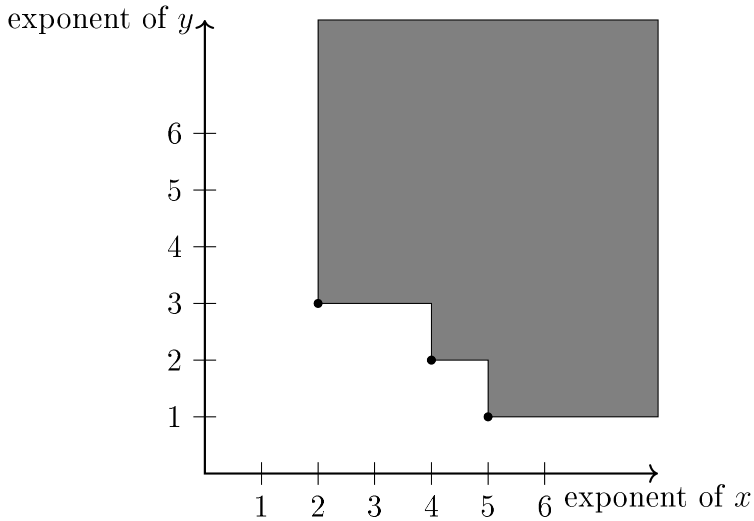
Case 2: I **is not artinian**

Suppose I is a monomial ideal in S that is not artinian. Then, as in Case 1, we can find a minimal set of monomials $m_1,\ldots,m_r$ that generate I, where each $m_j = x^{\alpha_j}y^{\beta_j}$ for some $\alpha_j,\beta_j \in \mathbb{N}$, not both zero since $I \neq S$. Similarly, we can form the set

$$A = \{(\alpha_j,\beta_j) \mid m_j = x^{\alpha_j}y^{\beta_j}, j=1,\ldots,r\}$$

and visualize I in $\mathbb{R}^2$ using the staircase formed by the points in A. Since I is not artinian, this staircase will not "touch" both the x and y axes. Therefore, S/I will now contain infinitely many unit squares.

Example 26. Let $I = \langle x^2y^3, x^4y^2, x^5y\rangle \subset S$. Then $A = \{(2,3),(4,2),(5,1)\}$ and we can visualize I in $\mathbb{R}^2$ as the shaded region in the graph below which extends upward and to the right.



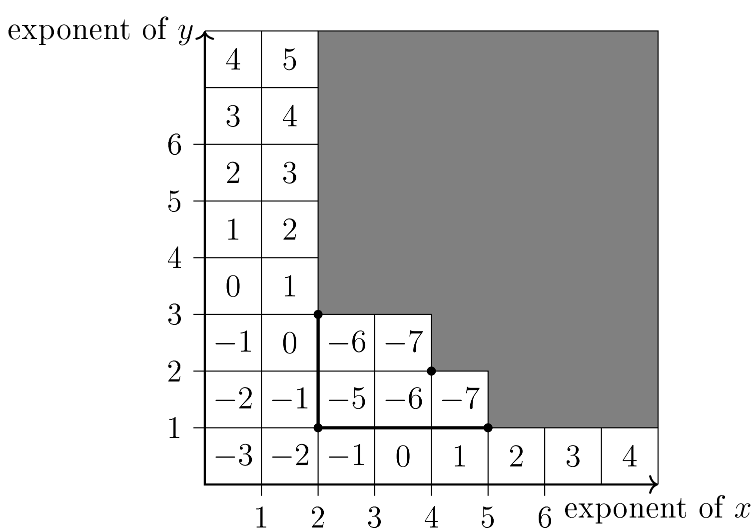
In order to relate this case to the artinian case, let

$$\alpha = \min\{\alpha_j\} \quad \text{and} \quad \beta = \min\{\beta_j\}.$$

Then drawing vertical and horizontal lines upwards and to the right of the point (α, β) produces an illustration that looks like and artinian ideal staircase with the origin shifted to (α, β). This also breaks up S/I into two sections: a finite set of unit squares in $\langle x^\alpha y^\beta \rangle / I$ and infinite rows and columns of unit squares not in $\langle x^\alpha y^\beta \rangle$. We will assign values to each of these unit squares depending on which section of S/I they are in.

> If a unit square is in $\langle x^\alpha y^\beta \rangle / I$ and has upper-right corner (s, t), then it is assigned the value $d = -(s+t)$, exactly as in Case 1. If a unit square in S/I is not in $\langle x^\alpha y^\beta \rangle$ and has lower-left corner (s, t), then it is assigned the value $e = s + t - (\alpha + \beta)$.

For the previous example, we have $(\alpha, \beta) = (2, 1)$ and our graph is as follows.



We are now able to compute $\operatorname{Ext}^i_S(S/I, S)$ by utilizing this setup.

Proposition 27. *Let I be a monomial ideal in $S = k[x,y]$ that is not artinian. As above, write $I = \langle m_1, \dots, m_r \rangle$ with $m_j = x^{\alpha_j} y^{\beta_j}$ for $j = 1, \dots, r$, and let $\alpha = \min\{\alpha_j\}, \beta = \min\{\beta_j\}$. Let $d_1, \dots, d_n$ denote the values of the unit squares in S/I contained in $\langle x^\alpha y^\beta \rangle / I$, and let $e_1, e_2, \dots$ denote the values of the unit squares in S/I not contained in $\langle x^\alpha y^\beta \rangle / I$. Then we have*

$$\operatorname{Ext}^i_S(S/I, S) = \begin{cases} k(-d_1) \oplus k(-d_2) \oplus \cdots \oplus k(-d_n), & i = 2 \\ k(-e_1) \oplus k(-e_2) \oplus \cdots, & i = 1 \\ 0, & i \neq 1, 2. \end{cases}$$

Proof. We proceed by induction on the number of unit squares in $\langle x^\alpha y^\beta \rangle / I$.
If there are no unit squares, then $I = \langle x^\alpha y^\beta \rangle$, and we need to show that

$$\operatorname{Ext}^i_S(S/I, S) = \begin{cases} k(-e_1) \oplus k(-e_2) \oplus \cdots, & i = 1 \\ 0, & i \neq 1. \end{cases}$$

First, we find $\operatorname{Ext}^i_S(I, S)$, viewing I as an S-module. The complex given by

$$0 \longrightarrow S(-(\alpha+\beta)) \xrightarrow{\varphi} I \longrightarrow 0 \quad \text{with} \quad \varphi : 1 \mapsto x^\alpha y^\beta$$

is exact, so we have $I \cong S(-(\alpha+\beta))$ as an S-module. Therefore,

$$\operatorname{Ext}^i_S(I, S) = \operatorname{Ext}^i_S(S(-(\alpha+\beta)), S) = \begin{cases} S(\alpha+\beta), & i = 0 \\ 0, & i \neq 0. \end{cases}$$

Then the exact complex $0 \longrightarrow I \longrightarrow S \longrightarrow S/I \longrightarrow 0$ gives the long exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Ext}^0_S(S/I, S) &\longrightarrow \operatorname{Ext}^0_S(S, S) \longrightarrow \operatorname{Ext}^0_S(I, S) \\ &\longrightarrow \operatorname{Ext}^1_S(S/I, S) \longrightarrow \operatorname{Ext}^1_S(S, S) \longrightarrow \operatorname{Ext}^1_S(I, S) \\ &\longrightarrow \operatorname{Ext}^2_S(S/I, S) \longrightarrow \operatorname{Ext}^2_S(S, S) \longrightarrow \operatorname{Ext}^2_S(I, S) \longrightarrow 0. \end{aligned}$$

This long exact sequence simplifies to

$$0 \longrightarrow \operatorname{Ext}^0_S(S/I, S) \xrightarrow{\varphi} S \xrightarrow{\psi} S(\alpha+\beta) \longrightarrow \operatorname{Ext}^1_S(S/I, S) \longrightarrow 0 \longrightarrow 0 \longrightarrow \operatorname{Ext}^2_S(S/I, S) \longrightarrow 0.$$

Then $\operatorname{Ext}^0_S(S/I, S) \cong \operatorname{Hom}_S(S/I, S) = 0$ since $I \neq 0$, and exactness of this sequence implies that $\operatorname{Ext}^2_S(S/I, S) = 0$. Therefore, this sequence further reduces to

$$0 \xrightarrow{\varphi} S \xrightarrow{\psi} S(\alpha+\beta) \xrightarrow{\phi} \operatorname{Ext}^1_S(S/I, S) \longrightarrow 0.$$

Exactness implies that ϕ is surjective and gives

$$\operatorname{Ext}^1_S(S/I, S) \cong S(\alpha+\beta)/\ker\phi = S(\alpha+\beta)/\operatorname{im}\psi = S(\alpha+\beta)/I.$$

Thus,

$$\operatorname{Ext}^i_S(S/I,S)=\begin{cases} S(\alpha+\beta)/I, & i=1\\ 0, & i\neq 1.\end{cases}$$

Then $S(\alpha+\beta)/I$ is generated as a k-vector space by the monomials in $S(\alpha+\beta)$ not in I. This gives

$$\begin{aligned} S(\alpha+\beta)/I &\cong \bigoplus_{i\geq 0} k(\alpha+\beta)x^i+\bigoplus_{i\geq 0} k(\alpha+\beta)x^iy+\cdots+\bigoplus_{i\geq 0} k(\alpha+\beta)x^iy^{\beta-1}\\ &\quad+\bigoplus_{j\geq\beta} k(\alpha+\beta)y^j+\bigoplus_{j\geq\beta} k(\alpha+\beta)xy^j+\cdots+\bigoplus_{j\geq\beta} k(\alpha+\beta)x^{\alpha-1}y^j\\ &\cong \bigoplus_{i\geq 0} k(\alpha+\beta-i)+\bigoplus_{i\geq 0} k(\alpha+\beta-(i+1))+\cdots+\bigoplus_{i\geq 1} k(\alpha+\beta-(i+\beta-1))\\ &\quad+\bigoplus_{j\geq\beta} k(\alpha+\beta-j)+\bigoplus_{j\geq\beta} k(\alpha+\beta-(j+1))+\cdots+\bigoplus_{j\geq\beta} k(\alpha+\beta-(j+\alpha-1)).\end{aligned}$$

Here we've used that in $k(\alpha+\beta)$ we have $\deg(1)=-(\alpha+\beta)$, which gives $\deg(x^iy^j)=-(\alpha+\beta)+i+j$. Thus, $k(\alpha+\beta)x^iy^j$ is generated in degree $\alpha+\beta-i-j$. Observe that each term in the resulting expression for $S(\alpha+\beta)/I$ is a copy of k generated in some degree $e=s+t-(\alpha+\beta)$ which is equal to the value assigned to a unit square in S/I. This gives a one to one correspondence between copies of k in $\operatorname{Ext}^1_S(S/I,S)$ and unit squares in S/I. Since the values of these squares are $e_1,e_2,\ldots$ we have

$$\operatorname{Ext}^i_S(S/I,S)=\begin{cases} k(-e_1)\oplus k(-e_2)\oplus\cdots, & i=1\\ 0, & i\neq 1\end{cases}$$

which completes the proof of the base case. Now suppose the proposition holds for monomial ideals I that are not artinian and have $n-1$ unit squares in $\langle x^\alpha y^\beta\rangle/I$.

For the inductive step, let I be a monomial ideal that is not artinian that has n unit squares in $\langle x^\alpha y^\beta\rangle/I$ with values given by $d_1,\ldots,d_n$. Recall that $d_j=-(s_j+t_j)$ is the value given to the unit square with upper-right corner (s_j,t_j). Without loss of generality, suppose $s_n+t_n=\max\{s_j+t_j\}$. Consider the ideal J containing I and the unit square with value d_n. Then J satisfies the inductive assumption. The short exact sequence

$$0\longrightarrow S/J\longrightarrow S/I\longrightarrow J/I\longrightarrow 0$$

gives the long exact sequence in cohomology (Theorem 5)

$$\begin{aligned} 0\longrightarrow \operatorname{Ext}^0_S(J/I,S)&\longrightarrow \operatorname{Ext}^0_S(S/I,S)\longrightarrow \operatorname{Ext}^0_S(S/J,S)\\ &\longrightarrow \operatorname{Ext}^1_S(J/I,S)\longrightarrow \operatorname{Ext}^1_S(S/I,S)\longrightarrow \operatorname{Ext}^1_S(S/J,S)\\ &\qquad\longrightarrow \operatorname{Ext}^2_S(J/I,S)\longrightarrow \operatorname{Ext}^2_S(S/I,S)\longrightarrow \operatorname{Ext}^2_S(S/J,S)\longrightarrow 0.\end{aligned}$$

Then J/I contains only the unit square with value $d_n=-(s_n+t_n)$. Since this square has lower-left corner (s_n-1,t_n-1), we have that $J/I\cong kx^{s_n-1}y^{t_n-1}$ as a k-vector space. To find what J/I is as an S-modules, consider the homorphism $\varphi : S(-(s_n+t_n)+2) \longrightarrow kx^{s_n-1}y^{t_n-1}$ given by $\varphi : 1 \mapsto x^{s_n-1}y^{t_n-1}$. This map is surjective and has $\ker\varphi = \langle x, y\rangle = m$, so we have that $J/I \cong S(d_n + 2)/m$ as an S-module. Therefore,

$$\operatorname{Ext}^i_S(J/I, S) = \operatorname{Ext}^i_S(S(d_n + 2)/m, S) = \begin{cases} k(-d_n), & i = 2 \\ 0, & i \neq 2. \end{cases}$$

By the inductive assumption, we also have

$$\operatorname{Ext}^i_S(S/J, S) = \begin{cases} k(-d_1) \oplus k(-d_2) \oplus \cdots \oplus k(-d_{n-1}), & i = 2 \\ k(-e_1) \oplus k(-e_2) \oplus \cdots, & i = 1 \\ 0, & i \neq 1, 2. \end{cases}$$

Thus, the long exact sequence becomes

$$0 \longrightarrow 0 \longrightarrow \operatorname{Ext}^0_S(S/I, S) \longrightarrow 0 \longrightarrow 0 \longrightarrow \operatorname{Ext}^1_S(S/I, S) \longrightarrow k(-e_1) + k(-e_2) + \cdots$$
$$\xrightarrow{\varphi} k(-d_n) \longrightarrow \operatorname{Ext}^2_S(S/I, S) \longrightarrow k(-d_1) + k(-d_2) + \cdots + k(-d_{n-1}) \longrightarrow 0.$$

This implies that $\operatorname{Ext}^0_S(S/I, S) = 0$ and φ is the zero map because

$$-d_n = s_n + t_n > \alpha + \beta \geq -e_j \qquad \text{for all} \qquad j \in \mathbb{N}.$$

Then $\ker\varphi = k(-e_1) + k(-e_2) + \cdots$ and so exactness gives

$$\operatorname{Ext}^1_S(S/I, S) = k(-e_1) \oplus k(-e_2) \oplus \cdots$$

and

$$\operatorname{Ext}^2_S(S/I, S) = k(-d_1) \oplus k(-d_2) \oplus \cdots \oplus k(-d_{n-1}) \oplus k(-d_n)$$

as desired. □

If I is artinian, then Proposition 27 still holds and in fact reduces to Proposition 25. Therefore, Proposition 27 allows us to compute $\operatorname{Ext}^i_S(S/I, S)$ for all monomial ideals in $S = k[x, y]$.

5.2 In the Multivariate Polynomial Ring

Let $S = k[x_1, \dots, x_n]$ and let I be any artinian monomial ideal of S. Then the argument used in Proposition 25 can be generalized to this case. Let a minimal set of generators for I be given by

$$I = \langle x_1^{p_1}, x_2^{p_2}, \dots, x_n^{p_n}, m_1, \dots, m_r\rangle \qquad \text{for some} \qquad p_1, \dots, p_n, r \geq 0,\ m_j = x_1^{\alpha_{j,1}} \cdots x_n^{\alpha_{j,n}} \in S.$$

Suppose that $I \neq S$. Then each $p_i \geq 1$, and we can "visualize" the ideal I as a multi-dimensional staircase in $\mathbb{R}^n$ by plotting the exponent vectors of the minimal generators. Then S/I contains finitely many n-dimensional unit cubes, since I is artinian. To each of these n-cubes we assign a value in the following way.

> Let $(\alpha_1, \dots, \alpha_n)$ be the point of the n-cube in S/I with maximal distance from the origin. Then the value of that n-cube is $-(\alpha_1 + \cdots + \alpha_n)$.

Proposition 28. *Let I be an artinian monomial ideal in $S = k[x_1, \dots, x_n]$, and write $I = \langle x_1^{p_1}, x_2^{p_2}, \dots, x_n^{p_n}, m_1, \dots, m_r \rangle$ as above. Denote the values of each of the unit n-cubes in S/I by $d_1, \dots, d_\ell$. Then we have*

$$\operatorname{Ext}^i_S(S/I, S) = \begin{cases} k(-d_1) \oplus k(-d_2) \oplus \cdots \oplus k(-d_\ell), & i = n \\ 0, & i \neq n. \end{cases}$$

Therefore, each unit n-cube in S/I corresponds to a copy of k generated in degree d_j in $\operatorname{Ext}^n_S(S/I, S)$.

This proposition can be proved in precisely the same manner as Proposition 25 using induction on the number ℓ of unit n-cubes in S/I. Therefore, we now know how to compute $\operatorname{Ext}^i_S(S/I, S)$ for all artinian monomial ideals in any finite number of variables. In particular, we can compute $\operatorname{reg}(I)$. In the case that I is an artinian monomial ideal generated in degree n, computing the regularity of I determines whether I has a linear resolution.

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