Crofton Formulæ

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Introduction

Integral geometry aims to provide geometric descriptions of statistical invariants of various ensembles of geometric objects. For example, it studies the relationship between the expectation of random variables and geometric quantities such as length and area. The field of integral geometry is inseparable from the Buffon Needle problem. The problem asks given a floor with equally spaced parallel lines at distance d apart and a needle of length $\ell < d$, what is the probability the needle will land on a line.[1]

However, problems of this type reached an impasse in the paradoxes described by J. Bertrand. One such paradox is: given a unit circle what is the probability that the length of a random chord is greater than $\sqrt{3}$. If a random chord is determined by its midpoint described in rectangular coordinates, then the probability is $\frac{1}{4}$. If a random chord is determined by its midpoint described in polar coordinates, then the probability is $\frac{1}{2}$. If a random chord is determined by its end points described in polar coordinates, then the probability is $\frac{1}{3}$. This paradox was rectified when Poincaré suggested that defining the probability of a geometric event should be invariant with respect to the natural symmetry group of that particular geometric problem.

The Crofton formula which was first proved for curves in the plane by M. Crofton is a classical integral geometry result. The result by M. Crofton says given a curve C in the plane, consider the function $|C \cap L|$ where L is an affine line in the plane. Then

$$\int_{\mathbf{Graff}^1(\mathbb{R}^2)} |C \cap L| dL = 2\,\mathrm{length(C)},$$

where $\mathbf{Graff}^1(\mathbb{R}^2)$ is the set of affine lines in the plane and dL denotes a measure on $\mathbf{Graff}^1(\mathbb{R}^2)$ that is invariant under the group of rigid motions of the Euclidean plane. This paper explains generalizations of the classical Crofton formula. More precisely, we discuss the Crofton formula for a C^2-curve in $\mathbb{R}^N$ and a codimension 1 submanifold of $\mathbb{R}^N$. Note that in the special case $N = 2$, a curve in $\mathbb{R}^2$ is also a codimension 1 submanifold of $\mathbb{R}^2$.

The paper begins with a discussion of jacobians of differentiable maps between Riemann manifolds and describes several practical methods of computing them. Section 2 describes a key theorem to the proofs of the Crofton formula, namely the coarea formula. We present two versions of this formula: the Riemannian version and also a version for densities. In

[1]Special thanks to my advisor Professor Liviu Nicolaescu whose patient expertise was essential to my understanding of these Crofton Formulæ and through out the writing of my senior thesis.

Section 3 we describe basic geometric facts concerning the Grassmannian of affine hyperplanes which are needed in the proof of Theorem 4.1 which is the Crofton formula for a C^2-curve in $\mathbb{R}^N$. In Section 5 we describe a density on the space of affine lines that is invariant with respect to the action of the group of rigid motions of the ambient space. This density is needed in the statement and proof of Theorem 6, the Crofton formula for an $(N-1)$-dimensional submanifold of $\mathbb{R}^N$.

1. Jacobians of a Linear Map

Definition 1.1. Suppose that $\boldsymbol{U}$ and $\boldsymbol{V}$ are *Euclidean* spaces of dimensions $n+k$ and k respectively, where $k, n \geq 0$. Given a linear map $A : U \to V$, the quantity

$$J_A := \sqrt{\det AA^*}$$

is called the Jacobian of the linear map A. □

Example 1.2. (a) Suppose that $L : \boldsymbol{U} \to \mathbb{R}$ is a linear functional and $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is an orthonormal basis of $\boldsymbol{U}$. If $L_i = L(\boldsymbol{e}_i)$ then

$$J_L = \|L\| = \sqrt{L_1^2 + \cdots + L_n^2}.$$

(b) Let $A : \boldsymbol{U} \to \boldsymbol{V}$ be a linear map between two Euclidean spaces. Then $J_A = 0$ if and only if A is not surjective. Indeed $J_A = 0$ if and only if $\ker AA^* \neq 0$. Observe that $\ker AA^* = \ker A^* = R(A)^\perp$, where $R(A)$ denotes the range of A. □

Lemma 1.3. *Suppose that $\boldsymbol{U}$ and $\boldsymbol{V}$ are* Euclidean *spaces of dimension $n+k$ and k respectively, where $k, n \geq 0$, and $A : \boldsymbol{U} \to \boldsymbol{V}$ is a* surjective *linear map. Then there exist Euclidean coordinates $x^1, \dots, x^{n+k}$ on $\boldsymbol{U}$, Euclidean coordinates $y^1, \dots, y^k$ on $\boldsymbol{V}$ and positive numbers $\mu_1, \dots, \mu_k$ such that, in these coordinates the operator A is described by*

$$y^j = \mu_j x^j, \;\; 1 \leq j \leq k.$$

The numbers $\mu_1^2, \dots, \mu_k^2$ are the eigenvalues of the positive symmetric operator $AA^* : \boldsymbol{V} \to \boldsymbol{V}$ so that

$$\mu_1 \cdots \mu_k = \sqrt{\det AA^*} = J_A.$$

Proof. Let $\boldsymbol{W}$ denote the orthogonal complement of $\ker A$ in $\boldsymbol{U}$. Denote by A_0 the restriction of A to W so that $A_0 : \boldsymbol{W} \to \boldsymbol{V}$ is a linear isomorphism. Note that $\boldsymbol{W}$ coincides with the range of the adjoint operator $A^* : \boldsymbol{V} \to \boldsymbol{U}$ so that

$$A_0 A_0^* = AA^*.$$

We want to find a linear isometry $R : \boldsymbol{V} \to \boldsymbol{W}$ such that the operator

$$B = A_0 R : \boldsymbol{V} \to \boldsymbol{V}$$

is symmetric. Note that since R is an isometry we have $R^{-1} = R^*$. Moreover we have a commutative diagram

$$\begin{array}{ccc} \boldsymbol{W} & \xrightarrow{A_0} & \boldsymbol{V} \\ {\scriptstyle R}\uparrow & & \uparrow{\scriptstyle \mathbb{1}_{\boldsymbol{V}}} \\ \boldsymbol{V} & \xrightarrow[B]{} & \boldsymbol{V} \end{array}$$

Note that $A_0A^* : \boldsymbol{V} \to \boldsymbol{V}$ is positive and symmetric. We define

$$R := A_0^*(A_0A_0^*)^{-1/2} : \boldsymbol{V} \to \boldsymbol{W}.$$

Let us show that R is indeed an isometry. Indeed, for any $\boldsymbol{v} \in \boldsymbol{V}$ we have

$$(R\boldsymbol{v}, R\boldsymbol{v}) = \big(\, A_0^*(A_0A_0^*)^{-1/2}\boldsymbol{v},\, A_0^*(A_0A_0^*)^{-1/2}\boldsymbol{v}\,\big) = \big(\,(A_0A_0^*)^{-1/2}\boldsymbol{v},\, A_0A_0^*(A_0A_0^*)^{-1/2}\boldsymbol{v}\,\big)$$
$$= \big(\,(A_0A_0^*)^{-1/2}\boldsymbol{v},\, (A_0A_0^*)^{1/2}\boldsymbol{v}\,\big) = (\boldsymbol{v}, \boldsymbol{v}).$$

Clearly $A_0R = A_0A_0^*(A_0A_0^*)^{-1/2} = (A_0A_0^*)^{1/2}$ is symmetric. Now choose an orthonormal basis that diagonalizes B. Transport it via R to an orthonormal basis of $\boldsymbol{W}$. With respect to these bases of $\boldsymbol{W}$ and $\boldsymbol{V}$ the operator A is described by a diagonal matrix with entries consisting of the eigenvalues of $A_0R = (A_0A_0^*)^{1/2}$.

□

It is convenient to give a more explicit description of J_A. This relies on the concept of Gramm determinant. More precisely, given a collection of vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$ in an Euclidean space $\boldsymbol{U}$ we define their *Gramm determinant* (or *Grammian*) to be the quantity

$$\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k) := \det \mathcal{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k), \;\; \mathcal{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k) := \Big(\, (\boldsymbol{u}_i, \boldsymbol{u}_j)_{\boldsymbol{U}}\,\Big)_{1 \le i,j \le k},$$

where $(-,-)_{\boldsymbol{U}}$ denotes the inner product in $\boldsymbol{U}$. Geometrically, $\sqrt{\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k)}$ is the k-dimensional volume of the parallelipiped spanned by the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$,

$$P(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k) = \Big\{\, \sum_{j=1}^k t_j\boldsymbol{u}_j; \;\; t_j \in [0,1]\,\Big\}.$$

Equivalently

$$\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k) = \big(\, \boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_k, \boldsymbol{u}_1 \wedge \cdots \wedge \boldsymbol{u}_k\,\big)_{\Lambda^k\boldsymbol{U}}$$

where $(-,-)_{\Lambda^kU}$ denotes the inner product on $\Lambda^k\boldsymbol{U}$ induced by the inner product in $\boldsymbol{U}$. Note that $\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k) = 0$ iff the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$ are linearly dependent and

$$\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k) = 1$$

if the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$ form an orthonormal system.

Lemma 1.4. *Let $A : \boldsymbol{U} \to \boldsymbol{V}$ be as in Lemma 1.3. Fix a basis $\boldsymbol{f}_{k+1}, \dots, \boldsymbol{f}_{n+k}$ of $\boldsymbol{U}_0 :=$ $\ker A$ and vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k \in \boldsymbol{U}$ such that the vectors $A\boldsymbol{u}_1, \dots, A\boldsymbol{u}_k$ span $\boldsymbol{V}$. Then*

$$J_A^2 = \frac{\mathbb{G}(A\boldsymbol{u}_1, \dots A\boldsymbol{u}_k)\mathbb{G}(\boldsymbol{f}_{k+1}, \dots, \boldsymbol{f}_{n+k})}{\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k, \boldsymbol{f}_{k+1}, \dots, \boldsymbol{f}_{n+k})}. \tag{1.1}$$

Proof. We first prove the result when $\dim \boldsymbol{U} = \dim \boldsymbol{V}$. In this case the collection $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$ is a basis of $\boldsymbol{U}$. Fix an orthonormal basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k$ of $\boldsymbol{U}$ denote by $T : \boldsymbol{U} \to \boldsymbol{U}$ the linear operator

$$\boldsymbol{e}_j \mapsto \boldsymbol{u}_j.$$

Then

$$\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k) = \det T^*T,$$
$$\mathbb{G}(A\boldsymbol{u}_1, \dots A\boldsymbol{u}_k) = \det((AT)^*(AT)) = |\det T^*| \det AA^* |\det T| = J_A^2 \det TT^*.$$

To deal with the general case, we denote by P_0 the orthogonal projection onto $\boldsymbol{U}_0$. Now define

$$\widehat{A} : \boldsymbol{U} \to \widehat{\boldsymbol{V}} := \boldsymbol{V} \oplus \boldsymbol{U}_0, \;\; \boldsymbol{u} \mapsto A\boldsymbol{u} \oplus P_0\boldsymbol{u}.$$

we equip $\widehat{\boldsymbol{V}}$ with the product Euclidean structure.

Let us observe that

$$J_A = J_{\widehat{A}}.$$

Indeed, with respect to the direct sum decomposition $\widehat{\boldsymbol{V}} = \boldsymbol{V} \oplus \boldsymbol{U}_0$ the operator $\widehat{A}\widehat{A}^*$ has the block decomposition

$$\widehat{A}\widehat{A}^* = \left[\begin{array}{cc} AA^* & 0 \\ 0 & \mathbb{1}_{\boldsymbol{U}_0} \end{array}\right]$$

so that

$$\det \widehat{A}\widehat{A}^* = \det AA^*.$$

Observe that in $\Lambda^{k+n}(\boldsymbol{V} \oplus \boldsymbol{U}_0)$ we have the equality

$$\hat{A}\boldsymbol{u}_1 \wedge \cdots \hat{A}\boldsymbol{u}_k \wedge \boldsymbol{f}_{k+1} \wedge \cdots \wedge \boldsymbol{f}_{k+n} = A\boldsymbol{u}_1 \wedge \cdots A\boldsymbol{u}_k \wedge \boldsymbol{f}_{k+1} \wedge \cdots \wedge \boldsymbol{f}_{k+n}$$

so that

$$\mathbb{G}(\widehat{A}\boldsymbol{u}_1, \dots, \widehat{A}\boldsymbol{u}_k, \widehat{A}\boldsymbol{f}_{k+1}, \dots, \widehat{A}\boldsymbol{f}_{n+k}) = \mathbb{G}(\widehat{A}\boldsymbol{u}_1, \dots, \widehat{A}\boldsymbol{u}_k, \boldsymbol{f}_{k+1}, \dots, \boldsymbol{f}_{n+k})$$

$$\mathbb{G}(A\boldsymbol{u}_1, \dots, A\boldsymbol{u}_k, \boldsymbol{f}_{k+1}, \dots, \boldsymbol{f}_{n+k}) = \mathbb{G}(A\boldsymbol{u}_1, \dots A\boldsymbol{u}_k)\mathbb{G}(\boldsymbol{f}_{k+1}, \dots, \boldsymbol{f}_{n+k}).$$

Now apply the first part of the proof to deduce that

$$J_A^2 = J_{\widehat{A}}^2 = \frac{\mathbb{G}(\widehat{A}\boldsymbol{u}_1, \dots, \widehat{A}\boldsymbol{u}_k, \widehat{A}\boldsymbol{f}_{k+1}, \dots, \widehat{A}\boldsymbol{f}_{n+k})}{\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k, \boldsymbol{f}_{k+1}, \dots, \boldsymbol{f}_{n+k})} = \frac{\mathbb{G}(A\boldsymbol{u}_1, \dots A\boldsymbol{u}_k)\mathbb{G}(\boldsymbol{f}_{k+1}, \dots, \boldsymbol{f}_{n+k})}{\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k, \boldsymbol{f}_{k+1}, \dots, \boldsymbol{f}_{n+k})}.$$

□

Definition 1.5. Suppose g is a Riemannian metric on the smooth manifold M. The *volume density* defined by g is the density denoted by $|dV_g|$ which associates to each $e \in C^\infty(\det TM)$ the pointwise length

$$x \mapsto |e(x)|_g.$$

If $(U_\alpha, (x_\alpha^i))$ is an atlas of M, then on each U_α we have top degree forms

$$dx_\alpha := dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^m,$$

to which we associate the density $|dx_\alpha|$. In the coordinates (x_α^i) the metric g can be described as

$$g = \sum_{i,j} g_{\alpha;ij} dx_\alpha^i \otimes dx_\alpha^j.$$

We denote by $|g_\alpha|$ the determinant of the symmetric matrix $g_\alpha = (g_{\alpha;ij})_{1i,jm}$. Then the restriction of $|dV_g|$ to U_α has the description

$$|dV_g| = \sqrt{|g_\alpha|}|dx_\alpha|$$

Suppose now that X and Y are Reimannian manifolds of dimensions $n+k$ and respectively k, $n \geq 0$ equipped with Riemannian metrics g_X and g_Y. We denote by $|dV_X|$ and $|dV_Y|$ the volume densities induced by g_X and respectively g_Y.

Suppose that $F : X \to Y$ is a C^1-map such that for any $\boldsymbol{p} \in M$ the differential

$$D_{\boldsymbol{p}}F : T_{\boldsymbol{p}}X \to T_{F(\boldsymbol{p})}Y$$

is surjective. We denote by $J_F(\boldsymbol{p})$ the Jacobian of this map.

2. The Coarea Formula

To develop the coarea formula we need the concept of a Hausdorff measure. Suppose (X, d) is a separable metric space. Fix a nonnegative real number r. For any positive number δ and any set $S \subset X$ we set

$$H^r_\delta(S) := \frac{\boldsymbol{\omega}_r}{2^r} \inf \left\{ \sum_{j\geq 1} (\operatorname{diam} B_j)^r; \;\; S \subset \bigcup_{j\geq 1} B_j, \;\; \operatorname{diam} B_j < \delta \right\}.$$

Note that if

$$0 < \delta_0 < \delta_1 \Rightarrow H^r_{\delta_0}(S) \geq H^r_{\delta_1}(S).$$

Thus the limit

$$\lim_{\delta \searrow 0} H^r_\delta(S)$$

exists and we denote it by H^r. We should note for most values of r that $H^r(S)$ is either 0 or ∞; however, there is a unique r called the Hausdorff dimension where the limit is neither 0 nor ∞. We now choose r to be the Hausdorff dimension of S. The correspondence $S \mapsto H^r(S)$ is an outer measure satisfying the Caratheodory condition, [7, Chap.12]

$$\operatorname{dist}(S_1, S_2) > 0 \Rightarrow H^r(S_1 \cup S_2) = H^r(S_1) + H^r(S_2).$$

This implies, [7, Chap. 5], that any Borel set B is measurable with respect to H^r, i.e.,

$$H^r(Y) = H^r(Y \cap B) + H^r(Y \setminus B), \;\; \forall Y \subset X.$$

We denote by $\sigma_r(X)$ the set of H^r-measurable subsets of X and by $\mathcal{H}^r$, or $\mathcal{H}^r_X$ the restriction of H^r to $\sigma_r(X)$. The measure $\mathcal{H}^r$ is called the r-th Hausdorff measure.

Example 2.1. (a) If M is a C^1-manifold of dimension m equipped with a C^0- Riemann metric g that induces a metric space structure on M, then for any Borel set $B \subset M$ we have

$$\mathcal{H}^m_M(B) = \operatorname{vol}_g(B).$$

In particular, $\mathcal{H}^m_M$ coincides with the measure induced by the volume density determined by g.
(b) If M is a C^1-submanifold of dimension k of C^1 Riemann manifold X of dimension n, then

$$\mathcal{H}^k_X(M) = \operatorname{vol}(M),$$

where $\operatorname{vol}(M)$ denotes the volume of M with respect to the Riemann metric induced by the Riemann metric on X.
(c) If X, Y are locally compact metric spaces, $F : X \to Y$ is a Lipschitz map with Lipschitz constant $\leq L$, and $B \subset X$ is a Borel set, then $F(B)$ is $\mathcal{H}^r_Y$-measurable and

$$\mathcal{H}^r_Y\big(F(B)\big) \leq L^r \mathcal{H}^r_X(B).$$

For proofs of the above statements (a), (b), (c) we refer to [7, Chap 12]. □

Theorem 2.2 (Eilenberg inequality)**.** *Suppose (X, d_X) is a separable metric space and Y is a C^1 manifold of dimension k equipped with a C^0-Riemann metric g. Denote by $d_Y : Y \times Y \to \mathbb{R}$ the metric on Y induced by g. Let $F : X \to Y$ be a map satisfying the Lipschitz condition*

$$d_Y\big(F(x_1), F(x_2)\big) \leq L d_X(x_1, x_2), \;\; \forall x_1, x_2 \in X.$$

Then for any $m \geq k$ there exists a constant[2] $C(m,k) > 0$ *such that for any Borel set $A \subset X$ we have*

$$\int_Y^* \mathcal{H}_X^{m-k}\big(A \cap F^{-1}(y)\big) d\mathcal{H}^k(y) \leq C(m,k) L^k \mathcal{H}^m(A),$$

where $\int^*$ denotes the upper Lebesgue integral, therefore,

$$\int_Y^* \mathcal{H}_X^{m-k}\big(A \cap F^{-1}(y)\big) d\mathcal{H}^k(y)$$
$$= \inf \left\{ \int_Y \psi \, d\mathcal{H}^k(y);\ 0 \leq \mathcal{H}_X^{m-k}\big(A \cap F^{-1}(y)\big) \leq \psi \ \text{and} \ \psi \ \text{is} \ \mu - \text{measurable} \right\}.$$

□

For a proof of this inequality we refer to [1, §13.3] or [2, §5.2.1]. As explained in [2, §5.2.1], this inequality implies the following technical result.

Corollary 2.3. *Let $F : X \to Y$ be as in Theorem 2.2. Then for any $m \geq k$ and any Borel subset $A \subset X$ the map*

$$Y \ni y \mapsto \mathcal{H}_X^{m-k}\big(A \cap F^{-1}(y)\big) \in [0, \infty]$$

is $\mathcal{H}_Y^k$-measurable. □

Theorem 2.4 (The co-area formula)**.** *Suppose X and Y are connected, Riemann C^1-manifolds of dimensions $n+k$ and respectively k, $n \geq 0$. If $F : X \to Y$ is a C^1-map satisfying the Lipschitz condition*

$$d_Y\big(F(x_1), F(x_2)\big) \leq L d_X(x_1, x_2), \ \ \forall x_1, x_2 \in X,$$

then, for any $\mathcal{H}_X^{n+k}$-measurable subset $A \subset X$ we have

$$\int_A J_F(x) d\mathcal{H}_X^{n+k}(x) = \int_Y \mathcal{H}_M^n\big(A \cap F^{-1}(y)\big) d\mathcal{H}_Y^k(y). \tag{2.1}$$

Corollary 2.5. *Let X, Y and $F : X \to Y$ be as in Theorem 2.4. Then for any measurable function $\phi : X \to \mathbb{R}$ we have*

$$\int_X \phi(\boldsymbol{p}) |dV_X(\boldsymbol{p})| = \int_Y \left(\int_{F^{-1}(\boldsymbol{q})} \frac{\phi(\boldsymbol{p})}{J_F(\boldsymbol{p})} |dV_{F^{-1}(\boldsymbol{q})}(\boldsymbol{p})| \right) |dV_Y(\boldsymbol{q})|, \tag{2.2}$$

Proof. Apply (2.4) to $\varphi = \frac{\phi}{J_F}$. □

Corollary 2.6. *Suppose X is a C^1 manifold equipped with a C^1-metric g_X, and $f : X \to \mathbb{R}$ is a C^1 function. Then for any measurable function $\phi : X \to \mathbb{R}$ we have*

$$\int_X \phi(\boldsymbol{p}) |dV_X(\boldsymbol{p})| = \int_{\mathbb{R}} \left(\int_{\{f=t\}} \frac{\phi(\boldsymbol{p})}{|\nabla f(\boldsymbol{p})|} |dV_{f^{-1}(t)}(\boldsymbol{p})| \right) dt. \tag{2.3}$$

In particular, by setting $\phi = 1$ we deduce

$$\text{vol}\,(X) = \int_{\mathbb{R}} \left(\int_{\{f=t\}} \frac{1}{|\nabla f(\boldsymbol{p})|} |dV_{f^{-1}(t)}(\boldsymbol{p})| \right) dt. \tag{2.4}$$

[2]We can choose $C(m,k) = \frac{\boldsymbol{\omega}_{m-k} \boldsymbol{\omega}_k}{\boldsymbol{\omega}_m}$

□

Example 2.7. We want to show how to use (2.4) to compute $\boldsymbol{\sigma}_n$, the "area" of the unit sphere

$$S^n = \Big\{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^n;\ \sum_{j=0}^n x_j^2 = 1 \Big\}.$$

Consider $f : S^n \to \mathbb{R}$, $f(x_0, \dots, x_n) = x_0$. Let $\boldsymbol{p} \in S^n$ such that $f(\boldsymbol{p}) = x_0(\boldsymbol{p}) = t$. Denote by φ the angle between the radius $O\boldsymbol{p}$ and the x_0-axis. Note that

$$\cos\varphi = x_0 = t.$$

The gradient of f is the projection of ∂_{x_0} on the tangent plane $T_{\boldsymbol{p}}S^n$. We deduce that

$$|\nabla f(\boldsymbol{p})| = |\partial_{x_0}| \sin\varphi = (1-t^2)^{1/2}.$$

The level set $\{f = t\}$ is an $(n-1)$-dimensional sphere of radius $(1-t^2)^{1/2}$ and we deduce

$$\int_{\{f=t\}} \frac{1}{|\nabla f(\boldsymbol{p})|} |dV_{f^{-1}(t)}(\boldsymbol{p})| = (1-t^2)^{-1/2} \mathrm{vol}\,(f=t) = \boldsymbol{\sigma}_{n-1}(1-t^2)^{\frac{n-2}{2}}.$$

Hence

$$\boldsymbol{\sigma}_n = \boldsymbol{\sigma}_{n-1} \int_{-1}^1 (1-t^2)^{\frac{n-2}{2}} dt = 2\boldsymbol{\sigma}_{n-1} \int_0^1 (1-t^2)^{\frac{n-2}{2}} dt$$

$(t = \sqrt{s})$

$$= \boldsymbol{\sigma}_{n-1} \int_0^1 (1-s)^{\frac{n}{2}-1} s^{\frac{1}{2}-1} ds = B\Big(\frac{n}{2}, \frac{1}{2}\Big).$$

The integral

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx, \quad p, q > 0$$

was computed by Euler and Legendre who showed that (see [8, Sec. 12.4])

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Hence

$$\frac{\boldsymbol{\sigma}_n}{\boldsymbol{\sigma}_{n-1}} = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})}. \tag{2.5}$$

Using the equalities $\boldsymbol{\sigma}_0 = 2$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ we deduce

$$\boldsymbol{\sigma}_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}.$$

We can obtain easily $\boldsymbol{\omega}_n$, the volume of the unit n-dimensional ball,

$$\boldsymbol{\omega}_n = \frac{1}{n}\boldsymbol{\sigma}_{n-1} = \frac{\pi^{\frac{n}{2}}}{\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}. \tag{2.6}$$

□

Corollary 2.8. *Let $F : X \to Y$ be as in Theorem 2.4. Then for any nonnegative measurable function $\varphi : X \to \mathbb{R}$ we have*

$$\int_X \varphi(x) J_F(x) d\mathcal{H}_X^{n+k}(x) = \int_Y \left(\int_{F^{-1}(y)} \varphi(x) d\mathcal{H}_X^n(x) \right) d\mathcal{H}_Y^k(y). \tag{2.7}$$

Proof. By Theorem 2.4 the equality (2.7) is true when φ is the characteristic function of a measurable subset of X. By linearity, (2.7) is true for linear combinations of such functions. We now observe that for any measurable nonnegative function φ we can find a sequence of simple functions $(\varphi_\nu)_{\nu\geq 1}$ that converges increasingly and almost everywhere to φ. □

The above coarea formula implies a counterpart for the integrals of densities. For more details about densities on manifolds and operations with them we refer to [3, Sec. 3.4.1, 9.1.1].

Corollary 2.9. *Suppose X, Y are smooth manifolds of the same dimension n and $F : X \to Y$ is a smooth proper map. Then for any volume density μ_Y on Y we have*

$$\int_Y |F^{-1}(y)| d\mu_Y(y) = \int_X F^*\mu_X(x). \tag{2.8}$$

Proof. Fix metrics g_X on X and g_Y on Y. Then there exists a smooth function $\rho_Y : Y \to \mathbb{R}$ such that $\mu_Y = \rho_Y |dV_{g_Y}|$. Then

$$F^*\mu_Y = (\rho_Y \circ F) F^*|dV_{g_Y}| = (\rho_Y \circ F) J_F |dV_{g_X}|.$$

The equality (2.8) now follows from the coarea formula (2.2) applied to the function $\phi = (\rho_Y \circ F) J_F$. □

Suppose that X, Y are smooth manifolds and $F : X \to Y$ is a smooth submersion. We set $n = \dim Y$, $k = \dim X - \dim Y > 0$. Fix a positive density μ_Y on Y and a density μ_X on X. The pair of densities μ_X, μ_Y define for each $y \in Y$ a density $\frac{\mu_X}{F^*\mu_Y}$ on the fiber $F^{-1}(y)$. More precisely, given $x \in F^{-1}(y)$ and a basis $V_1, \dots, V_k$ of $T_x F^{-1}(y)$ we set

$$\frac{\mu_X}{F^*\mu_Y}(V_1, \dots, V_k) := \frac{\mu_X(V_1, \dots, V_k, H_1, \dots H_n)}{\mu_Y(F_* H_1, \dots, F_* H_n)},$$

where $H_1, \dots, H_n \in T_x X$ are any n vectors such that $V_1, \dots, V_k, H_1, \dots, H_n$ is a basis of $T_x X$. The coarea formula then implies

$$\int_X \mu_X = \int_Y \left(\int_{F^{-1}(y)} \frac{\mu_X}{F^*\mu_Y} \right) \mu_Y \tag{2.9}$$

3. Affine Hyperplanes

Suppose that $\boldsymbol{X}$ is an N-dimensional Euclidean space with inner product $(-,-)$ and norm $\|-\|$. Denote by $S(\boldsymbol{X})$ the unit sphere in $\boldsymbol{X}$, by $\mathbf{Graff}^1(\boldsymbol{X})$ the set of affine hyperplanes in $\boldsymbol{X}$, and by $\mathbb{RP}(\boldsymbol{X})$ the projective space of lines through the origin in $\boldsymbol{X}$. We have a well known $2:1$ covering map

$$S(\boldsymbol{X}) \ni \boldsymbol{n} \stackrel{\pi}{\mapsto} [\boldsymbol{n}] := \operatorname{span}(\boldsymbol{n}) \in \mathbb{RP}(\boldsymbol{X}).$$

For any $L \in \mathbf{Graff}^1(\boldsymbol{X})$ denote by $[L]^\perp$ the line through the origin perpendicular to L. The resulting map

$$\mathbf{Graff}^1(\boldsymbol{X}) \ni L \mapsto [L]^\perp \in \mathbb{RP}(\boldsymbol{X})$$

defines a structure of real line bundle $\mathbf{Graff}^1(\boldsymbol{X}) \to \mathbb{RP}(\boldsymbol{X})$ canonically isomorphic to the tautological line bundle $\mathcal{U}_1 \to \mathbb{RP}(\boldsymbol{X})$.

The double cover π induces a double cover

$$\tilde{\pi} : \mathbb{R} \times S(\boldsymbol{X}) \to \mathbf{Graff}^1(\boldsymbol{X}),$$

$$\mathbb{R} \times S(\boldsymbol{X}) \times \ni (\lambda, \boldsymbol{n}) \mapsto H_{\lambda,\boldsymbol{n}} \in \mathbf{Graff}^1(\boldsymbol{X}),$$

where $H_{\lambda,\boldsymbol{n}}$ is the hyperplane

$$H_{\lambda,\boldsymbol{n}} := \big\{ \boldsymbol{x} \in \boldsymbol{X};\;\; (\boldsymbol{n}, \boldsymbol{x}) = \lambda \big\}.$$

The map $\tilde{\pi}$ is equivariant with respect to the natural action of the orthogonal group $O(\boldsymbol{X})$ on $\mathbb{R} \times S(\boldsymbol{X})$ and $\mathbf{Graff}^1(\boldsymbol{X})$. More explicitly, we can view $\mathbf{Graff}^1(\boldsymbol{X})$ as the quotient of $\mathbb{R} \times S(\boldsymbol{X})$ modulo the action of the reflection

$$R : \mathbb{R} \times S(\boldsymbol{X}) \to \mathbb{R} \times S(\boldsymbol{X}), R(\lambda, \boldsymbol{n}) = (-\lambda, -\boldsymbol{n}). \tag{3.1}$$

This reflection preserves the natural product metric on $\mathbb{R} \times S(\boldsymbol{X})$ and thus we have a well defined Riemann metric on $\mathbf{Graff}^1(\boldsymbol{X})$, which we will denote by g.

4. The Crofton formula for Curves

Suppose that C is simple closed C^2-curve in $\boldsymbol{X}$ parametrized by arclength

$$[0, S] \ni s \mapsto \gamma(s), \;\; \|\gamma'(s)\| = 1, \;\; S := \text{length}\,(C).$$

For any affine hyperplane $H \in \mathbf{Graff}^1(\boldsymbol{X})$ we denote by $|C \cap H| \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ the cardinality of the intersection $C \cap H$

Theorem 4.1 (Crofton)**.** *The function*

$$\mathbf{Graff}^1(\boldsymbol{X}) \ni H \mapsto |H \cap C| \in \mathbb{R}$$

is measurable and

$$\int_{\mathbf{Graff}^1(\boldsymbol{X})} |H \cap C|\, |dV_g(H)| = \boldsymbol{\omega}_{N-1} \text{length}\,(C) = \frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N+1}{2})} \text{length}\,(C). \tag{4.1}$$

Before proving the theorem we need some preliminary.

Lemma 4.2. *The incidence variety*

$$\mathcal{I}(C) = \big\{ (\boldsymbol{p}, H) \in C \times \mathbf{Graff}^1(\boldsymbol{X});\;\; \boldsymbol{p} \in H \;\big\}.$$

is a submanifold of $C \times \mathbf{Graff}^1(\boldsymbol{X})$.

Proof. Consider the smooth map

$$C \times S(\boldsymbol{X}) \ni (\boldsymbol{p}, \boldsymbol{n}) \overset{G}{\mapsto} G(\boldsymbol{p}, \boldsymbol{n}) = t \in \mathbb{R},$$

where $G(\boldsymbol{p}, \boldsymbol{n})$ is the dot product of $\boldsymbol{p}$ and $\boldsymbol{n}$.

The graph of G,

$$\Gamma_G := \big\{ (\boldsymbol{p}, \boldsymbol{n}, t) \in [0, S] \times S(\boldsymbol{X}) \times \mathbb{R}\};\;\; G(\boldsymbol{p}, \boldsymbol{n}) = t \,\big\}$$

is a submanifold of $C \times S(\boldsymbol{X}) \times \mathbb{R}$ since G is a smooth map between manifolds. The involution

$$\tilde{R} : [0, S] \times \mathbb{R} \times S(\boldsymbol{X}) \to [0, S] \times \mathbb{R} \times S(\boldsymbol{X}),\ \ \tilde{R}(s, \lambda, \boldsymbol{n}) = (s, -\lambda, -\boldsymbol{n})$$

acts smoothly and freely on Γ_G and $\mathcal{I}(C)$ is the quotient of this action and thus it is a smooth manifold. □

Viewed as a submanifold of $C \times \mathbf{Graff}^1(\boldsymbol{X})$, the incidence variety is equipped with the metric $\hat{g}$ induced by the product metric on $C \times \mathbf{Graff}^1(\boldsymbol{X})$. Consider natural projections

$$\begin{array}{ccc} & \mathcal{I}(C) & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ C & & \mathbf{Graff}^1(\boldsymbol{X}) \end{array}$$

$$\alpha(\boldsymbol{p}, H) = \boldsymbol{p},\ \ \beta(\boldsymbol{p}, H) = H.$$

Note that the fiber $\beta^{-1}(H)$ can be identified with the set $H \cap C$. The fiber of α over $\boldsymbol{p} \in C$ can be identified with $\mathbf{Graff}^1(\boldsymbol{X}, \boldsymbol{p}) \subset \mathbf{Graff}^1(\boldsymbol{X})$, the space of affine hyperplanes through $\boldsymbol{p}$. This is the quotient of the $\mathbb{Z}/2$-action on the submanifold

$$F_{\boldsymbol{p}} = \big\{ (\lambda, \boldsymbol{n}) \in \mathbb{R} \times S(\boldsymbol{X});\ \ (\boldsymbol{n}, \boldsymbol{p}) = \lambda \big\},$$

generated by the reflection $(\lambda, \boldsymbol{n}) \to (-\lambda, -\boldsymbol{n})$. This is an isometry of $\mathbb{R} \times S(\boldsymbol{X})$ and the metric induced on the quotient $\mathbf{Graff}^1(\boldsymbol{X}, \boldsymbol{p})$ coincides with metric as a submanifold of $\mathbf{Graff}^1(\boldsymbol{X})$. Note that $F_{\boldsymbol{p}}$ is the graph of the function

$$h_{\boldsymbol{p}} : S(\boldsymbol{X}) \to \mathbb{R},\ \ h_{\boldsymbol{p}}(\boldsymbol{n}) = (\boldsymbol{n}, \boldsymbol{p}).$$

As such it is diffeomorphic to $S(\boldsymbol{X})$.

Lemma 4.3. *The projections α and β satisfy the Lipschitz condition.*

Proof. Let d_C, $d_{\mathbf{Graff}^1(\boldsymbol{X})}$, $d_{C\times\mathbf{Graff}^1(\boldsymbol{X})}$, and $d_{\mathcal{I}(C)}$ be the distance functions on C, $\mathbf{Graff}^1(\boldsymbol{X})$, $C \times \mathbf{Graff}^1(\boldsymbol{X})$, and $\mathcal{I}(C)$ respectively.

First we will consider α. Observe that

$$d_C(\alpha(p_1, H_1), \alpha(p_2, H_2)) \leq d_{C\times\mathbf{Graff}^1(\boldsymbol{X})}((p_1, H_1), (p_2, H_2)) \leq d_{\mathcal{I}(C)}((p_1, H_1), (p_2, H_2)).$$

The first inequality follows since $C \times \mathbf{Graff}^1(\boldsymbol{X})$ has the product metric and

$$d_C(p_1, p_2) = d_C(\alpha(p_1, H_1), \alpha(p_2, H_2)).$$

Next observe that

$$d_{C\times\mathbf{Graff}^1(\boldsymbol{X})}(p, q) = \\ \inf\{L(\omega); \omega : [0, 1] \to C \times \mathbf{Graff}^1(\boldsymbol{X}) \text{ piecewise smooth path joining p to q}\}$$

and

$$d_{\mathcal{I}(C)}(p, q) = \inf\{L(\omega); \omega : [0, 1] \to \mathcal{I}(C) \text{ piecewise smooth path joining p to q}\}$$

where $L(\omega)$ is the length of ω. Since

$$\{L(\omega); \omega : [0, 1] \to \mathcal{I}(C) \text{ piecewise smooth path joining p to q}\} \\ \subseteq \{L(\omega); \omega : [0, 1] \to C \times \mathbf{Graff}^1(\boldsymbol{X}) \text{ piecewise smooth path joining p to q}\}$$

we deduce

$$d_{C\times \mathbf{Graff}^1(\boldsymbol{X})}((p_1,H_1),(p_1,H_1)) \leq d_{\mathcal{I}(C)}((p_1,H_1),(p_2,H_2)).$$

The statement concerning β is proved in a similar fashion. □

We will now prove Theorem 4.1.

Proof. Using the coarea formula (Theorem 2.4) we deduce

$$\int_{\mathbf{Graff}^1(\boldsymbol{X})} |H\cap C|\, |dV_g(H)| = \int_{\mathcal{I}(C)} J_\beta |dV_{\hat{g}}| = \int_C \left(\int_{\mathbf{Graff}^1(\boldsymbol{X},\boldsymbol{p})} \frac{J_\beta}{J_\alpha} dV_{\boldsymbol{p}} \right) ds(\boldsymbol{p}), \tag{4.2}$$

where J_α, J_β are the Jacobians of α and respectively β, and $dV_{\boldsymbol{p}}$ denotes the volume along $\mathbf{Graff}^1(\boldsymbol{X},\boldsymbol{p})$ with respect to the metric induced by the metric $\hat{g}$ on $\mathcal{I}(C)$. Note that

$$\int_{\mathbf{Graff}^1(\boldsymbol{X},\boldsymbol{p})} \frac{J_\beta}{J_\alpha} dV_{\boldsymbol{p}} = \frac{1}{2}\int_{F_{\boldsymbol{p}}} \frac{J_\beta}{J_\alpha} dV_{F_{\boldsymbol{p}}}.$$

Hence

$$\int_{\mathbf{Graff}^1(\boldsymbol{X})} |H\cap C|\, dV_g(H)| = \int_C \left(\frac{1}{2}\int_{F_{\boldsymbol{p}}} \frac{J_\beta}{J_\alpha} dV_{F_{\boldsymbol{p}}} \right) ds(\boldsymbol{p}). \tag{4.3}$$

To compute the various quantities above we first pick a point

$$(\boldsymbol{p}_0, H_0) = (\gamma(s_0), H_{\lambda_0,\boldsymbol{n}_0}) \in \mathcal{I}(C).$$

The tangent space $T_{(\boldsymbol{p}_0,H_0)}\mathcal{I}(C)$ is spanned by the velocities at $t=0$ of smooth paths

$$\psi : (-1,1) \to \mathcal{I}(C),\ \ \psi(0) = (p_0, H_0).$$

Such a path ψ is described by three paths

$$s : \mathbb{R} \to [0,S],\ t \mapsto s(t),\ s(0) = s_0,$$

$$\lambda : \mathbb{R} \to \mathbb{R},\ t \mapsto \lambda(t),\ \lambda(0) = \lambda_0,$$

and

$$\boldsymbol{n} : \mathbb{R} S(\boldsymbol{X}),\ t \mapsto \boldsymbol{n}(t),\ \boldsymbol{n}(0) = \boldsymbol{n}_0$$

subject to the constraint

$$(\boldsymbol{n}(t), \gamma(s(t))) = \lambda(t).$$

Upon derivating we deduce

$$(\dot{\boldsymbol{n}}, \boldsymbol{p}_0) + \dot{s}\big(\, \boldsymbol{n}_0, \gamma'(s_0)\,\big) = \dot{\lambda}.$$

For simplicity we set $\boldsymbol{v}_0 := \gamma'(s_0) \in S(\boldsymbol{X})$ and define

$$c : S(\boldsymbol{X}) \to \mathbb{R},\ \ c(\boldsymbol{n}) = (\boldsymbol{n}, \boldsymbol{v}_0), \tag{4.4}$$

so the above constraint reads

$$(\dot{\boldsymbol{n}}, \boldsymbol{p}_0) + \dot{s}c(\boldsymbol{n}_0) = \dot{\lambda}.$$

Fix an orthonormal basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{N-1}$ of $T_{\boldsymbol{n}_0}S(\boldsymbol{X})$ so that

$$(\boldsymbol{n}_0, \boldsymbol{e}_1) = \cdots = (\boldsymbol{n}_0, \boldsymbol{e}_{N-1}) = 0.$$

Viewed as a subspace of

$$T_{\boldsymbol{p}_0}C \times T_{\lambda_0}\mathbb{R} \times T_{\boldsymbol{n}_0}S(\boldsymbol{X}) = \mathbb{R}\times\mathbb{R}\times \operatorname{span}(\boldsymbol{e}_1,\dots,\boldsymbol{e}_{N-1})$$

we see that $T_{(\boldsymbol{p}_0,H_0)}\mathcal{I}(C)$ is described by triplets

$$(\dot{s}, \dot{\lambda}, \dot{\boldsymbol{n}}) \in \mathbb{R}\times\mathbb{R}\times\boldsymbol{X},\ \ \dot{\boldsymbol{n}} \perp \boldsymbol{n}_0,$$

satisfying

$$\dot{\lambda} = (\dot{\boldsymbol{n}}, \boldsymbol{p}_0) + \dot{s}c(\boldsymbol{n}_0).$$

In particular we see that the component $\dot{\lambda}$ is uniquely determined by the components $\dot{s}, \dot{\boldsymbol{n}}$. We write

$$L(\dot{s}, \dot{\boldsymbol{n}}) := (\dot{\boldsymbol{n}}, \boldsymbol{p}_0) + c(\boldsymbol{n}_0)\dot{s}.$$

We obtain a basis of $T_{(\boldsymbol{p}_0, H_0)}\mathfrak{I}(C)$ consisting of the vectors

$$\boldsymbol{u}_1 := (0, L(0, \boldsymbol{e}_1), \boldsymbol{e}_1), \dots, \boldsymbol{u}_{N-1} := (0, L(0, \boldsymbol{e}_{N-1}), \boldsymbol{e}_{N-1}),$$
$$\boldsymbol{u}_N := (1, L(1,0), 0) = \big(1, c(\boldsymbol{n}_0), 0\big).$$

Note that the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{N-1}$ belong to the kernel of the differential of α at $(\boldsymbol{p}_0, H_0)$. On the other hand

$$\beta_*\boldsymbol{u}_1 = (L(0, \boldsymbol{e}_1), \boldsymbol{e}_1), \quad \beta_*\boldsymbol{u}_{N-1} = (L(0, \boldsymbol{e}_{N-1}), \boldsymbol{e}_{N-1}),$$
$$\beta_*\boldsymbol{u}_N = (L(1,0), 0) = (c(\boldsymbol{n}_0), 0).$$

For simplicity we set

$$L_k := L(0, \boldsymbol{e}_k) = (\boldsymbol{p}_0, \boldsymbol{e}_k), \quad k = 1, \dots, N-1.$$

For $1 \le j, k \le N-1$ we have

$$(\beta_*\boldsymbol{u}_j, \beta_*\boldsymbol{u}_k) = \begin{cases} L_j L_k, & j \neq k \\ 1 + L_k^2, & j = k \end{cases}, \quad (\beta_*\boldsymbol{u}_j, \beta_*\boldsymbol{u}_N) = c_0(\boldsymbol{n}_0)L_j,$$
$$(\beta_*\boldsymbol{u}_N, \beta_*\boldsymbol{u}_N) = c(\boldsymbol{n}_0)^2.$$

We deduce (writing for simplicity c instead of $c(\boldsymbol{n}_0)$)

$$\mathbb{G}(\beta_*\boldsymbol{u}_1, \dots, \beta_*\boldsymbol{u}_N) = \det \begin{bmatrix} 1 + L_1^2 & L_1L_2 & L_1L_3 & \cdots & L_1L_{N-1} & cL_1 \\ L_1L_2 & 1 + L_2^2 & L_2L_3 & \cdots & L_2L_{N-1} & cL_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ L_1L_{N-1} & L_2L_{N-1} & L_3L_{N-1} & \cdots & 1 + L_{N-1}^2 & cL_{N-1} \\ cL_1 & cL_2 & cL_3 & \cdots & cL_{N-1} & c^2 \end{bmatrix}$$
$$= c^2 \det \underbrace{\begin{bmatrix} 1 + L_1^2 & L_1L_2 & L_1L_3 & \cdots & L_1L_{N-1} & L_1 \\ L_1L_2 & 1 + L_2^2 & L_2L_3 & \cdots & L_2L_{N-1} & L_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ L_1L_{N-1} & L_2L_{N-1} & L_3L_{N-1} & \cdots & 1 + L_{N-1}^2 & L_{N-1} \\ L_1 & L_2 & L_3 & \cdots & L_{N-1} & 1 \end{bmatrix}}_{=:A}.$$

We have the following elementary fact whose proof we postpone to the end of this section.

$$\det A = 1. \tag{4.5}$$

We then see that

$$\mathbb{G}(\beta_*\boldsymbol{u}_1, \dots, \beta_*\boldsymbol{u}_N) = c^2.$$

Therefore, we see that at the point $(s_0, \lambda_0, \boldsymbol{n}_0) \in \mathfrak{I}(C)$ we have

$$\frac{J_\beta}{J_\alpha} = \frac{\sqrt{\mathbb{G}(\beta_*\boldsymbol{u}_1, \dots, \beta_*\boldsymbol{u}_N)}}{\sqrt{\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_{N-1})}} = \frac{|c(\boldsymbol{n}_0)|}{\sqrt{\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_{N-1})}}.$$

and that

$$\frac{1}{2}\int_{F_{\boldsymbol{p}}} \frac{J_\beta}{J_\alpha} dV_{F_{\boldsymbol{p}}} = \frac{1}{2}\int_{F_{\boldsymbol{p}}} \frac{|c(\boldsymbol{n})|}{\sqrt{\mathbb{G}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_{N-1})}} dV_{F_{\boldsymbol{p}}}. \tag{4.6}$$

Now we have that $F_{\boldsymbol{p}}$ is diffeomorphic to $S(\boldsymbol{X})$ by the following map:

$$\Phi : S(\boldsymbol{X}) \to F_{\boldsymbol{p}},\ \Phi(\boldsymbol{n}) = (h_{\boldsymbol{p}}(\boldsymbol{n}), \boldsymbol{n}).$$

The Jacobian J_Φ is

$$J_\Phi = \sqrt{\mathbb{G}(\Phi_* \boldsymbol{e}_1, \cdots, \Phi_* \boldsymbol{e}_{N-1})}.$$

where $\boldsymbol{e}_1,\dots,\boldsymbol{e}_{N-1}$ is the orthonormal basis of $T_{\boldsymbol{n}_0}S(\boldsymbol{X})$ defined above. We then note that for $1 \le j \le N-1$ we have $\Phi_*\boldsymbol{e}_j = (L(0,e_j), e_j)$. Thus, $J_\Phi = J_\alpha$.

By the change of variables under the diffeomorphism Φ and (4.4) we see

$$\frac{1}{2}\int_{F_{\boldsymbol{p}}} \frac{|c(\boldsymbol{n})|}{\sqrt{\mathbb{G}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_{N-1})}} dV_{F_{\boldsymbol{p}}} = \frac{1}{2}\int_{S(\boldsymbol{X})} |(\boldsymbol{n}, \boldsymbol{v}_0)| dV_{S(\boldsymbol{X})}(\boldsymbol{n}). \tag{4.7}$$

Choose an orthonormal basis $\mathbf{b}_1,\dots,\mathbf{b}_N$ of $\boldsymbol{X}$ such that $\mathbf{b}_N = \boldsymbol{v}_0$. Denote by $x_1,\dots,x_N$ the coordinates determined by this basis. Observe that for $(x_1,\dots,x_N) \in S(\boldsymbol{X})$ we have

$$c(x_1,\dots,x_N) = x_N.$$

Denote by $S(\boldsymbol{X})_+$ the upper hemisphere

$$S(\boldsymbol{X})_+ = S(\boldsymbol{X}) \cap \{x_N \ge 0\}.$$

We deduce that

$$\frac{1}{2}\int_{S(\boldsymbol{X})} |(\boldsymbol{n}, \boldsymbol{v}_0)| dV_{S(\boldsymbol{X})}(\boldsymbol{n}) = \int_{S(\boldsymbol{X})_+} (\boldsymbol{n}, \boldsymbol{v}_0) dV_{S(\boldsymbol{X})}(\boldsymbol{n}) = \int_{S(\boldsymbol{X})_+} c(\boldsymbol{n}) dV_{S(\boldsymbol{X})}(\boldsymbol{n}).$$

To compute the last integral we argue as in Example 2.7. The coarea formula (Theorem 2.4) shows that

$$\int_{S(\boldsymbol{X})_+} c(\boldsymbol{n}) dV_{S(\boldsymbol{X})}(\boldsymbol{n}) = \int_0^1 \left(\int_{c^{-1}(t)} \frac{t}{J_c} dV_{c^{-1}(t)} \right) dt. \tag{4.8}$$

Let $\boldsymbol{n} \in c^{-1}(t) \subset S(\boldsymbol{X})$. Observe that the Jacobian of c at $\boldsymbol{n}$ is $\|\nabla c(\boldsymbol{p})\|$. Denote by φ the angle between $\boldsymbol{n}$ and $\mathbf{b}_N$ so that

$$\cos\varphi = (\boldsymbol{n}, \mathbf{b}_N) = c(\boldsymbol{n}) = t.$$

The gradient of c is the projection on $T_{\boldsymbol{n}}S(\boldsymbol{X})$ of the vector $\mathbf{b}_N$. More precisely we have

$$\mathbf{b}_N = (\boldsymbol{n}, \mathbf{b}_N)\boldsymbol{n} + \nabla c(\mathbf{b}_N) = t\mathbf{b}_N + \nabla c(\mathbf{b}_N).$$

Pythagoras' Theorem implies

$$J_c(\boldsymbol{n}) = \|\nabla c(\boldsymbol{n})\| = \sqrt{1-t^2}.$$

This shows that along $c^{-1}(t)$ we have

$$\frac{t}{J_c} = t(1-t^2)^{-1/2}.$$

Using this in (4.8) we deduce

$$\int_{S(\boldsymbol{X})_+} c(\boldsymbol{n}) dV_{S(\boldsymbol{X})}(\boldsymbol{n}) = \int_0^1 \left(\int_{c^{-1}(t)} dV_{c^{-1}(t)} \right) t(1-t^2)^{-1/2} dt \tag{4.9}$$

The fiber $c^{-1}(t)$ is an $(N-2)$-dimensional round sphere of radius $\sqrt{1-t^2}$. Hence

$$\int_{c^{-1}(t)} dV_{c^{-1}(t)} = (1-t^2)^{\frac{N-2}{2}}\boldsymbol{\sigma}_{N-2},$$

where $\boldsymbol{\sigma}_k$ denotes the "area" of the round unit k-dimensional sphere. Substituting this in (4.9) we deduce

$$\int_{S(\boldsymbol{X})_+} c(\boldsymbol{n})dV_{S(\boldsymbol{X})}(\boldsymbol{n}) = \boldsymbol{\sigma}_{N-2}\int_0^1 t(1-t^2)^{\frac{N-3}{2}}dt$$

$$\overset{t\to\sqrt{u}}{=} \frac{\boldsymbol{\sigma}_{N-2}}{2}\int_0^1 (1-u)^{\frac{N-3}{2}}du = \frac{\boldsymbol{\sigma}_{N-2}}{2}B\left(1,\frac{N-1}{2}\right),$$

where we recall that $B(x,y)$ is the Beta function

$$B(x,y) := \int_0^1 s^{x-1}(1-s)^{y-1}ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ \ x,y>0.$$

We deduce that

$$\frac{1}{2}\int_{S(\boldsymbol{X})}|c(\boldsymbol{n})|dV_{S(\boldsymbol{X})}(\boldsymbol{n}) = \int_{S(\boldsymbol{X})_+} c(\boldsymbol{n})dV_{S(\boldsymbol{X})}(\boldsymbol{n}) = \frac{\boldsymbol{\sigma}_{N-2}}{2}\frac{\Gamma(1)\Gamma(\frac{N-1}{2})}{\Gamma(\frac{N+1}{2})}$$

$$= \frac{\boldsymbol{\sigma}_{N-2}}{2\cdot\frac{N-1}{2}} = \frac{\boldsymbol{\sigma}_{N-2}}{N-1} = \boldsymbol{\omega}_{N-1},$$

where $\boldsymbol{\omega}_k$ denotes the volume of the unit Euclidean k-dimensional ball. Using this in (4.6) and (4.7) we deduce

$$\int_{\mathbf{Graff}^1(\boldsymbol{X})}|H\cap C|\,|dV_g(H)| = \boldsymbol{\omega}_{N-1}\int_C ds(\boldsymbol{p}) = \boldsymbol{\omega}_{N-1}\,\mathrm{length}\,(C) \overset{(2.6)}{=} \frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N+1}{2})}\,\mathrm{length}\,(C).$$

□

Proof of (4.5) We compute $\det A$ by computing the eigenvalues of A. Note first that

$$A = 1+B,\ \ B = \begin{bmatrix} L_1^2 & L_1L_2 & L_1L_3 & \cdots & L_1L_{N-1} & L_1\\ L_1L_2 & L_2^2 & L_2L_3 & \cdots & L_2L_{N-1} & L_2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ L_1L_{N-1} & L_2L_{N-1} & L_3L_{N-1} & \cdots & L_{N-1}^2 & L_{N-1}\\ L_1 & L_2 & L_3 & \cdots & L_{N-1} & 0 \end{bmatrix}. \tag{4.10}$$

It turns out that that it is easier to compute the eigenvalues of B. Let $C_1,\dots,C_N$ be the columns of matrix B. Therefore,

$$B = \begin{bmatrix} C_1 & C_2 & C_3 & \cdots & C_N \end{bmatrix}.$$

Note that $C_1 = \frac{L_1}{L_i}C_i$ for all $i\in\{1,2,\dots,N-1\}$. Thus, we see that

$$\boldsymbol{w}_{i-1} := \boldsymbol{f}_1 - \frac{L_1}{L_i}\boldsymbol{f}_i \in \ker B,\ \ \forall i\in\{2,\dots,N-1\},$$

where $\boldsymbol{f}_i \in \mathbb{R}^N$ is the vector

$$\boldsymbol{f}_i = \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \vdots \\ \delta_{iN} \end{bmatrix}$$

where (δ_{ij}) is the Kronecker symbol. First we compute $B\boldsymbol{f}_1$ and $B\boldsymbol{f}_N$. We set

$$T := \sum_{i=2}^{N-1} L_i^2$$

and we deduce

$$B\boldsymbol{f}_1 = -\sum_{i=2}^{N-1} L_i^2 \boldsymbol{w}_{i-1} + T\boldsymbol{f}_1 + L_1\boldsymbol{f}_N, \quad B\boldsymbol{f}_N = \frac{1}{L_1}\left(-\sum_{i=2}^{N-1} L_i^2 \boldsymbol{w}_{i-1} + T\boldsymbol{f}_1\right).$$

We note that $(\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{N-2}, B\boldsymbol{f}_1, B\boldsymbol{f}_N)$ is a basis. Now we want to write B in this basis. Therefore we note,

$$\forall i \in \{2, \ldots, N-1\}, \quad B\boldsymbol{w}_{i-1} = 0$$

and

$$B(B\boldsymbol{f}_1) = L_1 B\boldsymbol{f}_N + TB\boldsymbol{f}_1 \text{ and } B(B\boldsymbol{f}_N) = \frac{T}{L_1} B\boldsymbol{f}_1.$$

Thus B is the block diagonal matrix.

$$B = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{B} \end{bmatrix},$$

where

$$\tilde{B} = \begin{bmatrix} T & \frac{T}{L_1} \\ L_1 & 0 \end{bmatrix}.$$

The eigenvalues of B are

$$\mu_1 = \cdots = \mu_{N-2}, = 0, \quad \mu_{N-1}, \; \mu_N$$

where μ_{N-1}, μ_N are the eigenvalues for the 2×2 matrix $\tilde{B}$. First note that

$$\det \tilde{B} = \mu_{N-1}\mu_N = -T \text{ and } \operatorname{tr} \tilde{B} = \mu_{N-1} + \mu_N = T.$$

We have

$$\det A = \det(1 + B) = \prod_{j=1}^{n} (1 + \mu_j) = (1 + \mu_{N-1})(1 + \mu_N)$$

$$= 1 + \mu_{N-1} + \mu_N + \mu_{N-1}\mu_N = 1 + \operatorname{tr} \tilde{B} + \det \tilde{B} = 1.$$

□

5. Affine Lines

Let $\mathbf{Graff}^{N-1}(\boldsymbol{X})$ be the set of affine lines in $\boldsymbol{X}$. Consider the tautological vector bundle $\mathcal{U}_1 \to \mathbb{RP}(\boldsymbol{X})$. This is a subbundle of the trivial vector bundle $V = X \times \mathbb{RP}(\boldsymbol{X}) \to \mathbb{RP}(\boldsymbol{X})$. V is equipped with the natural metric and we denote by $\mathcal{U}_1^\perp \to \mathbb{RP}(\boldsymbol{X})$ the orthogonal complement of $\mathcal{U}_1$ in V. The fiber of $\mathcal{U}_1^\perp$ over $L \in \mathbb{RP}(\boldsymbol{X})$ is canonically identified with the orthogonal complement $L^\perp$ of $L \in V$. The points of $\mathcal{U}_1^\perp$ are the pairs $(\boldsymbol{q}, L)$ where $L \in \mathbb{RP}(\boldsymbol{X})$ and $\boldsymbol{q} \in L^\perp$. Consider the following map

$$\mathbf{Graff}^{N-1}(\boldsymbol{X}) \ni L \mapsto [L] \in \mathbb{RP}(\boldsymbol{X}), \tag{5.1}$$

where $[L] \in \mathbb{RP}(\boldsymbol{X})$ denotes the affine subspace through the origin parallel to L. This map defines a structure of a real line bundle $\mathbf{Graff}^{N-1}(\boldsymbol{X}) \to \mathbb{RP}(\boldsymbol{X})$ that is canonically isomorphic to to the line bundle $\mathcal{U}_1^\perp \to \mathbb{RP}(\boldsymbol{X})$.

In the sequel, for clarity we will denote by $\bullet$ the inner product in $\boldsymbol{X}$. Define

$$P(\boldsymbol{X}) := \big\{\, (\boldsymbol{n}, \boldsymbol{q}) \in S(\boldsymbol{X}) \times \boldsymbol{X};\;\; \boldsymbol{q} \perp \boldsymbol{n} \,\big\}.$$

Note that $P(\boldsymbol{X})$ can be identified with the total space of the tangent bundle of the sphere $S(\boldsymbol{X})$. Note that

$$T_{(\boldsymbol{q}_0, \boldsymbol{n}_0)} P(\boldsymbol{X}) = \big\{\, (\dot{\boldsymbol{q}}, \dot{\boldsymbol{n}}) \in \boldsymbol{X} \times \boldsymbol{X};\;\; \dot{\boldsymbol{q}} \bullet \boldsymbol{n}_0 + \boldsymbol{q}_0 \bullet \dot{\boldsymbol{n}} = \dot{\boldsymbol{n}} \bullet \boldsymbol{n}_0 = 0 \,\big\}.$$

Denote by p the natural projection $P(\boldsymbol{X}) \to S(\boldsymbol{X})$. Note that $p^{-1}(\boldsymbol{n}) = \boldsymbol{n}^\perp$. We have a natural density $d\hat{\mu}$ on $P(\boldsymbol{X})$ uniquely determined by the requirement

$$\int_{P(\boldsymbol{X})} f(\boldsymbol{n}, \boldsymbol{q}) d\hat{\mu}(\boldsymbol{n}, \boldsymbol{q}) = \int_{S(\boldsymbol{X})} \left(\int_{\boldsymbol{n}^\perp} f(\boldsymbol{n}, \boldsymbol{q}) dV_{\boldsymbol{n}^\perp}(\boldsymbol{q}) \right) dV_{S(\boldsymbol{X})}(\boldsymbol{n}),$$

where $f : P(\boldsymbol{X}) \to \mathbb{R}$ is a compactly supported continuous function, $dV_{\boldsymbol{n}^\perp}$ is the Euclidean volume form on $\boldsymbol{n}^\perp$ and $dV_{S(\boldsymbol{X})}$ is the natural Euclidean volume form on $S(\boldsymbol{X})$.

The double cover π of $\mathbb{RP}(\boldsymbol{X})$ induces the double cover

$$\hat{\pi} : P(\boldsymbol{X}) \to \mathbf{Graff}^{N-1}(\boldsymbol{X}),\;\; P(\boldsymbol{X}) \ni (\boldsymbol{q}, \boldsymbol{n}) \mapsto L_{\boldsymbol{q},\boldsymbol{n}} \in \mathbf{Graff}^{N-1}(\boldsymbol{X}),$$

where $L_{\boldsymbol{q},\boldsymbol{n}}$ is the affine line $L_{\boldsymbol{q},\boldsymbol{n}} := \boldsymbol{q} + \operatorname{span}(\boldsymbol{n})$. The map $\hat{\pi}$ is equivariant with respect to the natural action of the orthogonal group $O(\boldsymbol{X})$ on $P(\boldsymbol{X})$

We can view $\mathbf{Graff}^{N-1}(\boldsymbol{X})$ as the quotient of $P(\boldsymbol{X})$ modulo the action of the reflection

$$\hat{R} : P(\boldsymbol{X}) \to P(\boldsymbol{X}),\;\; \hat{R}(\boldsymbol{q}, \boldsymbol{n}) = (\boldsymbol{q}, -\boldsymbol{n}). \tag{5.2}$$

This reflection preserves the density $d\hat{\mu}$ on $P(\boldsymbol{X})$ and thus we have a well defined density on $\mathbf{Graff}^{N-1}(\boldsymbol{X})$, which we will denote by $d\hat{\mu}$.

Let us present a local description of $d\hat{\mu}$. Fix a point $(\boldsymbol{n}_0, \boldsymbol{q}_0) \in S(\boldsymbol{X})$ and a local moving orthonormal frame $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_N)$ of $\boldsymbol{X}$ in a neighborhood U of $\boldsymbol{n}_0$ such that

$$\boldsymbol{e}_N(\boldsymbol{n}) = \boldsymbol{n},\;\; \forall \boldsymbol{n} \in S(\boldsymbol{X}) \cap U.$$

Fix a neighborhood V of $\boldsymbol{q}_0$ in $\boldsymbol{X}$. Then

$$d\hat{\mu} = \left| \left(\bigwedge_{i=1}^{N-1} d\boldsymbol{q} \bullet \boldsymbol{e}_i \right) \wedge \left(\bigwedge_{j=1}^{N-1} d\boldsymbol{n} \bullet \boldsymbol{e}_i \right) \right|.$$

The above description shows that $d\hat{\mu}$ coincides with the density on $\mathbf{Graff}^{N-1}(\boldsymbol{X})$ described in [5, Sec.II.12.2]. As explained there, this density is invariant with respect to the action on $\mathbf{Graff}^{N-1}(\boldsymbol{X})$ of the group of rigid motions of $\boldsymbol{X}$.

6. The Crofton Formula for Hypersurfaces

Suppose that M is $(N-1)$-dimensional submanifold of $\boldsymbol{X}$. For any affine line $L \in \mathbf{Graff}^{N-1}(\boldsymbol{X})$ we denote by $|L \cap M| \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ the cardinality of the intersection $L \cap M$.

Theorem 6.1 (Crofton)**.** *The function*

$$\mathbf{Graff}^{N-1}(\boldsymbol{X}) \ni L \mapsto |L \cap M| \in \mathbb{R}$$

is measurable and

$$\int_{\mathbf{Graff}^{N-1}(\boldsymbol{X})} |L \cap M| \, |d\hat{\mu}(L)| = \frac{\pi^{\frac{N}{2}} \Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2}) \Gamma(\frac{N+1}{2})} \operatorname{vol}_{N-1}(M). \tag{6.1}$$

Proof. Consider the incidence variety

$$\mathcal{I}(M) := \big\{ (\boldsymbol{p}, L) \in M \times \mathbf{Graff}^{N-1}(\boldsymbol{X}); \;\; \boldsymbol{p} \in L \,\big\}.$$

Define

$$\hat{\mathcal{I}}(M) := \big\{ (\boldsymbol{p}, \boldsymbol{n}, \boldsymbol{q}) \in M \times P(\boldsymbol{X}); \;\; \boldsymbol{q} = \operatorname{Proj}^{\perp}_{\boldsymbol{n}} \boldsymbol{p} \,\big\},$$

where $\operatorname{Proj}_{\boldsymbol{n}^\perp} \boldsymbol{p} = \boldsymbol{p} - (\boldsymbol{p} \bullet \boldsymbol{n}) \boldsymbol{n}$ is the orthogonal projection onto the orthogonal complement of $\operatorname{span}(\boldsymbol{n})$.

We have a $2:1$ covering map $\hat{\mathcal{I}}(M) \to \mathcal{I}(M)$ that sends $(\boldsymbol{p}, \boldsymbol{n}, \operatorname{Proj}^{\perp}_{\boldsymbol{n}} \boldsymbol{p})$ to the pair $(\boldsymbol{p}, L_{\boldsymbol{q},\boldsymbol{n}}) \in \mathcal{I}(M)$, $\boldsymbol{q} = \operatorname{Proj}_{\boldsymbol{n}^\perp} \boldsymbol{p}$. In fact $\mathcal{I}(M)$ is the quotient of $\hat{\mathcal{I}}(M)$ with respect to the action of the reflection

$$(\boldsymbol{p}, \boldsymbol{n}, \boldsymbol{q}) \leftrightarrow (\boldsymbol{p}, -\boldsymbol{n}, \boldsymbol{q}). \tag{6.2}$$

The space $\hat{\mathcal{I}}(M)$ is a manifold since it is the graph of the map

$$\operatorname{Proj} : M \times S(\boldsymbol{X}) \to \boldsymbol{X}, \;\; \operatorname{Proj}(\boldsymbol{p}, \boldsymbol{n}) = \operatorname{Proj}_{\boldsymbol{n}^\perp} \boldsymbol{p}.$$

We deduce that the incidence variety $\mathcal{I}(M)$ is a submanifold of $M \times \mathbf{Graff}^{N-1}(\boldsymbol{X})$.

Consider natural projections

$$\begin{array}{ccc} & \mathcal{I}(M) & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ M & & \mathbf{Graff}^{N-1}(\boldsymbol{X}), \end{array} \;\; , \;\; \alpha(\boldsymbol{p}, L) = \boldsymbol{p}, \;\; \beta(\boldsymbol{p}, L) = L.$$

Note that the fiber $\beta^{-1}(L)$ can be identified with the set $L \cap M$. The fiber of α over $\boldsymbol{p} \in M$ can be identified with $\mathbf{Graff}^{N-1}(\boldsymbol{X}, \boldsymbol{p}) \subset \mathbf{Graff}^{N-1}(\boldsymbol{X})$, the space of affine lines through $\boldsymbol{p}$.

Note that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{I}(M) & \xleftarrow{2:1} & \hat{\mathcal{I}}(M) \\ \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \hat{\beta}} \\ \mathbf{Graff}^{N-1}(\boldsymbol{X}) & \xleftarrow[2:1]{} & P(\boldsymbol{X}) \end{array}$$

where $\hat{\beta}(\boldsymbol{p}, \boldsymbol{n}, \boldsymbol{q}) = (\boldsymbol{n}, \boldsymbol{q})$, $\boldsymbol{q} = \operatorname{Proj}_{\boldsymbol{n}^\perp} \boldsymbol{p}$. We denote by $\hat{\alpha}$ the composition

$$\hat{\mathcal{I}}(M) \xrightarrow{2:1} \mathcal{I}(M) \xrightarrow{\alpha} M.$$

We deduce

$$\int_{\mathbf{Graff}^{N-1}(\boldsymbol{X})} |L\cap M|\,|d\hat{\mu}(L)| = \frac{1}{2}\int_{P(\boldsymbol{X})} |L_{\boldsymbol{q},\boldsymbol{n}}\cap M|\,|d\hat{\mu}(\boldsymbol{n},\boldsymbol{q})|.$$

We deduce

$$\int_{P(\boldsymbol{X})} |L_{\boldsymbol{q},\boldsymbol{n}}\cap M|\,|d\hat{\mu}(\boldsymbol{n},\boldsymbol{q})| \stackrel{(2.8)}{=} \int_{\hat{\jmath}(M)} \hat{\beta}^* d\hat{\mu}(\boldsymbol{n},\boldsymbol{q}),$$
$$\int_{\mathbf{Graff}^{N-1}(\boldsymbol{X})} |L\cap M|\,d\hat{\mu}(L) = \frac{1}{2}\int_{\hat{\jmath}(M)} \hat{\beta}^* d\hat{\mu}(\boldsymbol{n},\boldsymbol{q}).$$

Using (2.9) we deduce

$$\int_{\hat{\jmath}(M)} \hat{\beta}^* d\hat{\mu}(\boldsymbol{n},\boldsymbol{q}) = \int_M \left(\int_{\hat{\alpha}^{-1}(\boldsymbol{p})} \frac{\hat{\beta}^* d\hat{\mu}}{\hat{\alpha}^* dV_M}\right) dV_M(\boldsymbol{p}),$$

where dV_M is the volume density induced by the natural metric on M. We conclude that

$$\int_{\mathbf{Graff}^{N-1}(\boldsymbol{X})} |L\cap M|\,d\hat{\mu}(L) = \frac{1}{2}\int_M \left(\int_{\hat{\alpha}^{-1}(\boldsymbol{p})} \frac{\hat{\beta}^* d\hat{\mu}}{\hat{\alpha}^* dV_M}\right) dV_M(\boldsymbol{p}). \tag{6.3}$$

Let us compute

$$\int_{\hat{\alpha}^{-1}(\boldsymbol{p})} \frac{\hat{\beta}^* \mu}{\hat{\alpha}^* dV_M}$$

We have

$$\hat{\beta}(\boldsymbol{p},\boldsymbol{n},\operatorname{Proj}_{\boldsymbol{n}^\perp}\boldsymbol{p}) = (\boldsymbol{n},\operatorname{Proj}_{\boldsymbol{n}^\perp}\boldsymbol{p}) = (\boldsymbol{n},\boldsymbol{p}-(\boldsymbol{p}\bullet\boldsymbol{n})\boldsymbol{n})$$
$$\hat{\beta}^* d\hat{\mu} = \left|\left(\bigwedge_{i=1}^{N-1} d\Big((\boldsymbol{p}-(\boldsymbol{p}\bullet\boldsymbol{n})\boldsymbol{n})\bullet\boldsymbol{e}_i\Big)\right)\wedge\left(\bigwedge_{j=1}^{N-1} d\boldsymbol{n}\bullet\boldsymbol{e}_j\right)\right|$$
$$= \left|\left(\bigwedge_{i=1}^{N-1} (d\boldsymbol{p}-(\boldsymbol{p}\bullet\boldsymbol{n})d\boldsymbol{n})\bullet\boldsymbol{e}_i\right)\wedge\left(\bigwedge_{j=1}^{N-1} d\boldsymbol{n}\bullet\boldsymbol{e}_j\right)\right|$$
$$= \left|\left(\bigwedge_{i=1}^{N-1} d\boldsymbol{p}\bullet\boldsymbol{e}_i\right)\wedge\left(\bigwedge_{j=1}^{N-1} d\boldsymbol{n}\bullet\boldsymbol{e}_j\right)\right|$$

Fix a point $\boldsymbol{p}_0\in M$. Note that

$$\hat{\alpha}^{-1}(\boldsymbol{p}_0) = \big\{\,(\boldsymbol{p}_0,\boldsymbol{n},\operatorname{Proj}_{\boldsymbol{n}^\perp}\boldsymbol{p}_0);\;\; \boldsymbol{n}\in S(\boldsymbol{X})\,\big\}.$$

Note that we have a natural diffeomorphism

$$\Phi_{\boldsymbol{p}_0}: S(\boldsymbol{X})\to\hat{\alpha}^{-1}(\boldsymbol{p}_0),\;\; \boldsymbol{n}\mapsto\Phi_{\boldsymbol{p}_0}(\boldsymbol{n}) = (\boldsymbol{p}_0,\boldsymbol{n},\operatorname{Proj}_{\boldsymbol{n}^\perp}\boldsymbol{p}_0).$$

Fix an orthonormal frame $(\boldsymbol{u}_1,\dots,\boldsymbol{u}_{N-1},\boldsymbol{u}_N)$ of $\boldsymbol{X}$ near $\boldsymbol{p}_0$ such that $\boldsymbol{u}_N$ is a unit normal vector field along M. We denote this unit normal vector vield by $\boldsymbol{\nu}(\boldsymbol{p})$ so $\boldsymbol{\nu}(\boldsymbol{p})^\perp = T_{\boldsymbol{p}}M$. Note that the hyperplane $\boldsymbol{n}^\perp$ intersects the hyperplane $T_{\boldsymbol{p}_0}M$ transversally if and only if $\boldsymbol{n}\neq\pm\boldsymbol{\nu}(\boldsymbol{p}_0)$. Denote by $F_{\boldsymbol{p}_0}$ the fiber $\alpha^{-1}(\boldsymbol{p}_0)$ with the points $\Phi_{\boldsymbol{p}_0}(\pm\boldsymbol{\nu}(\boldsymbol{p}_0))$ removed. We have

$$\int_{\hat{\alpha}^{-1}(\boldsymbol{p}_0)} \frac{\hat{\beta}^* d\hat{\mu}}{\hat{\alpha}^* dV_M} = \int_{F_{\boldsymbol{p}_0}} \frac{\hat{\beta}^* d\hat{\mu}}{\hat{\alpha}^* dV_M} \tag{6.4}$$

Fix a point $(\boldsymbol{p}_0,\boldsymbol{n}_0,\boldsymbol{q}_0)\in F_{\boldsymbol{p}_0}$, $\boldsymbol{q}_0 = \operatorname{Proj}_{\boldsymbol{n}_0^\perp}\boldsymbol{p}_0$, $\boldsymbol{n}_0\neq\pm\boldsymbol{\nu}(\boldsymbol{p}_0)$. Next, fix an orthonormal frame $\boldsymbol{e}_1,\dots,\boldsymbol{e}_N$ of $\boldsymbol{X}$ along $S(\boldsymbol{X})$ near $\boldsymbol{n}_0$ such that $\boldsymbol{n}_0 = \boldsymbol{e}_N(\boldsymbol{n}_0)$.

The hyperplanes $\boldsymbol{n}(\boldsymbol{p}_0)^\perp$ and $\boldsymbol{n}_0^\perp$ intersect transversally along a subspace of dimension $N-2$. We can assume that we have fixed the frames $\boldsymbol{u}_i$, $\boldsymbol{e}_j$ so that $\boldsymbol{e}_j(\boldsymbol{n}_0)=\boldsymbol{u}_j(\boldsymbol{p}_0)$, $\forall j=1,\dots,N-2$, i.e., $\boldsymbol{e}_1(\boldsymbol{n}_0),\dots,\boldsymbol{e}_{N-2}(\boldsymbol{p}_0)$ is a basis of $\boldsymbol{\nu}(\boldsymbol{p}_0)^\perp\cap\boldsymbol{n}_0^\perp$. Set

$$\hat{\boldsymbol{u}}_i := (\boldsymbol{u}_i, 0, \operatorname{Proj}_{\boldsymbol{n}_0}\boldsymbol{u}_i)\in T_{\boldsymbol{p}_0,\boldsymbol{n}_0,\boldsymbol{q}_0}\hat{\mathfrak{I}}(M),\ \ i=1,\dots,N-1.$$

Note that

$$\hat{\beta}_*(\hat{\boldsymbol{u}}_i)=\boldsymbol{u}_i,\ \ dV_M(\boldsymbol{u}_1,\dots,\boldsymbol{u}_{N-1})=1.$$

Now observe that

$$\left(\bigwedge_{i=1}^{N-1} d\boldsymbol{p}\bullet\boldsymbol{e}_i\right)\big(\,\hat{\boldsymbol{u}}_1,\dots,\hat{\boldsymbol{u}}_{N-1}\,\big)\big|_{\boldsymbol{p}_0,\boldsymbol{n}_0,\boldsymbol{q}_0}=\det\Big(\boldsymbol{u}_i(\boldsymbol{n}_0)\bullet\boldsymbol{e}_j(\boldsymbol{n}_0)\Big)_{1\leq i,j\leq N-1}$$

$$=\boldsymbol{u}_{N-1}(\boldsymbol{p}_0)\bullet\boldsymbol{e}_{N-1}(\boldsymbol{n}_0)=\cos\angle\big(\,\boldsymbol{u}_{N-1}(\boldsymbol{p}_0),\boldsymbol{e}_{N-1}(\boldsymbol{n}_0)\,\big).$$

On the other hand (see Figure 1 for an explanation)

$$\cos\angle\big(\,\boldsymbol{u}_{N-1}(\boldsymbol{p}_0),\boldsymbol{e}_{N-1}(\boldsymbol{n}_0)\,\big)=\sin\angle\big(\,\boldsymbol{\nu}(\boldsymbol{p}_0),\boldsymbol{n}_0\,\big)$$

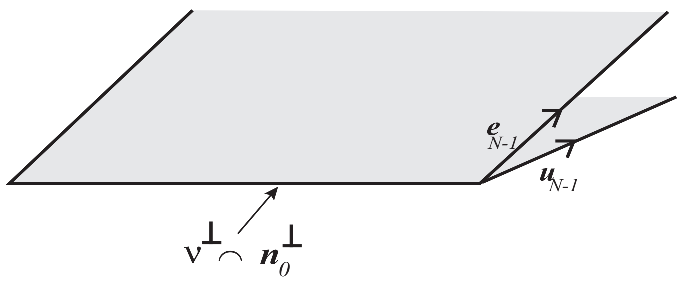

Figure 1. *Planes intersecting transversally.*

We deduce that

$$\frac{\beta^* d\hat{\mu}}{\hat{\alpha}^* dV_M}\Big|_{\boldsymbol{p}_0,\boldsymbol{n}_0,\boldsymbol{q}_0}=\sin\angle\big(\,\boldsymbol{\nu}(\boldsymbol{p}_0),\boldsymbol{n}_0\,\big)\underbrace{\left|\left(\bigwedge_{j=1}^{N-1} d\boldsymbol{n}\bullet\boldsymbol{e}_i\right)\right|}_{=:\rho}\Big|_{\boldsymbol{n}_0}$$

We deduce that

$$\int_{F_{\boldsymbol{p}_0}}\frac{\beta^* d\hat{\mu}}{\hat{\alpha}^* dV_M}=\int_{S(\boldsymbol{X})}\sin\angle\big(\,\boldsymbol{\nu}(\boldsymbol{p}_0),\boldsymbol{n}\,\big)\Phi^*_{\boldsymbol{p}_0}\rho(\boldsymbol{n})$$

Now observe that the density on $S(\boldsymbol{X})$ is none other than the volume density defined by the round metric so

$$\int_{F_{\boldsymbol{p}_0}}\frac{\beta^*\mu}{\hat{\alpha}^* dV_M}=\underbrace{\int_{S(\boldsymbol{X})}\sin\angle\big(\,\boldsymbol{\nu}_0,\boldsymbol{n}\,\big)\,dV_{S(\boldsymbol{X})}(\boldsymbol{n})}_{=:I_N(\boldsymbol{\nu}_0)},\ \ \boldsymbol{\nu}_0=\boldsymbol{\nu}(\boldsymbol{p}_0).\tag{6.5}$$

Due to the rotational symmetry of the sphere, the integral $I_N(\boldsymbol{\nu}_0)$ is independent of $\boldsymbol{\nu}_0$ so it is a number that depends only on the dimension N. More precisely we have the following lemma.

Lemma 6.2. *Let $\boldsymbol{\nu} \in S(\boldsymbol{X})$. Then*

$$I_N = \int_{S(\boldsymbol{X})} \sin\angle\big(\boldsymbol{\nu}, \boldsymbol{n}\big)\, dV_{S(\boldsymbol{X})}(\boldsymbol{n}) = 2\frac{\boldsymbol{\sigma}_{N-2}\boldsymbol{\sigma}_N}{\boldsymbol{\sigma}_{N-1}} = \frac{2\pi^{n/2}\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})\Gamma(\frac{N+1}{2})}.$$

Proof. Fix an orthonormal basis $\mathbf{b}_1, \dots, \mathbf{b}_N$ of $\boldsymbol{X}$ such that $\mathbf{b}_N = \boldsymbol{\nu}$. Now let

$$f : S(\boldsymbol{X}) \to [-1, 1], \quad f(\boldsymbol{n}) = \cos\angle\big(\boldsymbol{\nu}, \boldsymbol{n}\big).$$

Concretely, $f(x_1, \dots, x_N) = x_N$, $\forall (x_1, \dots, x_N) \in S(\boldsymbol{X})$, where $(x_1, \dots, x_N)$ denote the Euclidean coordinates determined by the basis $\mathbf{b}_1, \dots, \mathbf{b}_N$. Then by the coarea formula we have

$$I_N = \int_{S(\boldsymbol{X})} \sin\angle\big(\boldsymbol{\nu}, \boldsymbol{n}\big) dV_{S(\boldsymbol{X})}(\boldsymbol{n}) = \int_{-1}^{1} \left(\int_{f^{-1}(t)} \frac{\sqrt{1-t^2}}{J_f} dV_{f^{-1}(t)} \right) dt \tag{6.6}$$

As shown in Example 2.7

$$J_f(\boldsymbol{n}) = \|\nabla f(\boldsymbol{n})\| = \sqrt{1-t^2}, \quad t = f(\boldsymbol{n}).$$

From (6.6) we deduce

$$I_N = \int_{-1}^{1} \left(\int_{f^{-1}(t)} dV_{f^{-1}(t)} \right) dt = \int_{-1}^{1} \big(1-t^2\big)^{\frac{N-2}{2}} \boldsymbol{\sigma}_{N-2} dt = 2\boldsymbol{\sigma}_{N-2} \int_0^1 \big(1-t^2\big)^{\frac{N-2}{2}} dt$$

$$\stackrel{t \to \sqrt{u}}{=} \boldsymbol{\sigma}_{N-2} \int_0^1 (1-u)^{\frac{N}{2}-1} u^{\frac{1}{2}-1} du = \boldsymbol{\sigma}_{N-2} B\left(\frac{1}{2}, \frac{N}{2}\right) = \boldsymbol{\sigma}_{N-2} \frac{\Gamma(\frac{N}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{N+1}{2})}$$

$$\stackrel{(2.5)}{=} \frac{\boldsymbol{\sigma}_{N-2}\boldsymbol{\sigma}_N}{\boldsymbol{\sigma}_{N-1}} = \frac{2\pi^{n/2}\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})\Gamma(\frac{N+1}{2})}.$$

□

From Lemma 6.2 and equations (6.3), (6.4), and (6.5) we conclude

$$\int_{\mathbf{Graff}^{N-1}(\boldsymbol{X})} |L \cap M|\, d\mu(L) = \frac{I_N}{2} \int_M dV_M(\boldsymbol{p}) = \frac{\pi^{N/2}\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})\Gamma(\frac{N+1}{2})} \operatorname{vol}_{N-1}(M).$$

□

Remark 6.3. (a) Note that, when $N = 2$, a curve in $\mathbb{R}^2$ is also a codimension 1 submanifold of $\mathbb{R}^2$. In this case the constants that appear in the Crofton formulæ (4.1) and (6.1) coincide

$$\frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N+1}{2})} = \frac{\pi^{n/2}\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{n+1}{2})} = 2.$$

(b) When $N = 3$ the constant that appears in the Crofton formula (4.1) is

$$\frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N+1}{2})} = \pi$$

and the constant that appears in the Crofton formula (6.1) is

$$\frac{\pi^{N/2}\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})\Gamma(\frac{n+1}{2})} = \frac{\pi^2}{2}.$$

□

References

[1] Y.D. Burago, V.A. Zalgaller: *Geometric Inequalities*, Springer Verlag, 1988.
[2] S.G. Krantz, H. R. Parks: *Geometric Integration Theory*, Birkhäuser, 2008.
[3] L.I. Nicolaescu: *Lectures on the Geometry of Manifolds*, 2nd Edition World Scientific, 2007.
[4] L.I. Nicolaescu: The coarea formula, notes for a seminar.
[5] L. Santaló: *Integral Geometry and Geometric Probability*, Cambridge University Press, 2004.
[6] L. Simon: *Lectures on Geometric Measure Theory*, Proc. Centre. Math. Anal., vol. 3, Australian National University, Canberra, 1983.
[7] M. E. Taylor: *Measure Theory and Integration*, Amer. Math. Soc., 2006.
[8] E.T. Whittaker, G.N. Watson: *A Course of Modern Analysis*, 4th Edition, Cambridge University Press, 1950.

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